ЗАЛОГ

FERMAT'S LAST THEOREM

ITS HISTORY AND THE NATURE OF THE KNOWN RESULTS CONCERNING IT

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1. Introduction. Fermat's famous theorem [13b], dating from about the year 1637, that

$$x^n + y^n = z^n \tag{1}$$

has no solution in integers x, y, and z, none zero, if n is an integer greater than 2, still awaits a proof or disproof.* Euler, Legendre, Gauss, Abel, Cauchy, Dirichlet, Lamé, Kummer, Frobenius, Furtwängler, and Dickson are among the well-known mathematicians who have worked on the problem, at least for particular values of n. A number of investigators have obtained criteria for the solution of (1) independent of x, y, and z. In other words they have shown that (1) is impossible for a certain class of exponents n. We shall be interested here in examining the nature of these criteria in the cases where it may be shown that none of the classes referred to is the null class. Also, many individuals have given criteria for (1) which cannot, apparently, be applied to any particular exponent in a finite number of steps. No conditions of this latter character will be considered here.†

Fermat [13c] left a proof for $n=4$; and for $n=3$ Euler [13a] gave a proof in the year 1770. The latter proof is incomplete but the missing steps were supplied by later writers.‡

* Concerning the theorem, Kronecker, Vorlesungen über Zahlentheorie, p. 23, expresses himself as follows: "Mit diesem Satze . . . haben sich die Mathematiker . . . vielleicht mehr beschäftigt, als mit irgend einem andern, und wohl keiner hat, abgesehen etwa von der Quadratur des Kreises, zu so vielen falschen und irrtümlichen Deductionen Veranlassung gegeben."

† Many criteria of this latter character have been given in connection with the theorem. In particular, in recent years Morishima, Krasner (particularly interesting is [24]), and the writer have obtained such results. For the history of the problem prior to the year 1927 *cf.* [12] and [64]. Expositions of some of the principal results are given in [2], [11], [21], [32], [39]. The Fermat relation (2) has been generalized in various ways, but none of them will be considered here. Referring to the exposition of some of the theory of algebraic numbers given in this paper, it may be noted that the simplest method for developing this subject in general is based on a theory of semi-groups and groups. However, we shall not use such ideas here, as it would be necessary to develop them to some length before we could begin the present exposition and this would require more space than we have at our disposal. Of course, if we would proceed from the group standpoint, the results in this subject would enable us to derive a great deal more of the theory of algebraic numbers than we consider here. We give few proofs in full. Our idea is to bring out some of the extraordinary features in the proofs, for example, the method of infinite descent. Usually, we give references to the literature which will enable the reader to locate complete proofs of results stated. Not a great deal of the theory of algebraic numbers or of analysis is needed to follow the proof of the statements in this paper (excepting Theorem IV) until we reach Theorem X. But then the situation is different. The argument needed to derive the class number of the cyclotomic field requires considerable analysis and the complete proofs of the Theorems X to XIV inclusive, are all, unfortunately, complicated and require a great deal of the theory of algebraic numbers.

‡ A complete proof from Euler's point of view is not very simple. *cf.* Uspensky [50], pp. 394–5, 408–11.

To prove Fermat's statement in general, it is then sufficient to demonstrate the impossibility of

$$x^l + y^l + z^l = 0 \tag{2}$$

in non-zero integers x, y, and z, for any odd prime $l>3$. This was proved by Legendre [33] in 1823 for $l=5$ and in 1839 by Lamé [30] for $l=7$. Stimulated by these results a number of mathematicians of the time made many efforts to extend the methods of proof to other values of l. For example, Cauchy [7] published a number of results concerning (2), as did Lamé, based on assumptions which were later proved false. The study of these mistakes led Kummer to the notion of *ideal numbers* which we shall discuss later.

The condition that (2) be satisfied in integers may also be expressed in the form that if r is a proper rational fraction then $\sqrt[l]{1+r^l}$ is rational.* The problem seems never to have been successfully attacked, in any special case, by using this form of (2). All known methods begin by taking the equation (2) and assuming that integers x, y, and z satisfy it. The properties of these integers are then investigated. It is important to make use of the fact that x, y, and z are integers since our theorem does not hold if we substitute for integers, quantities in certain more general algebraic systems. For example, if $a+b=c$ then

$$(\sqrt[n]{a})^n + (\sqrt[n]{b})^n = (\sqrt[n]{c})^n.$$

Now in particular, if in (2), x, y, and z are prime to each other and to l, this condition *will be referred to as Case* I *of Fermat's last theorem*; if x, y, and z are prime to each other and one of them is divisible by l, the condition *will be called Case* II *of the theorem*. So far many more classes of criteria have been given for Case I than Case II and for simplicity we shall now consider the former type of criteria. *But first we must signalize a very important theorem of arithmetic stating that a composite integer decomposes uniquely into prime factors,*† as this result and its generalizations have apparently played a fundamental role in all contributions which have been made to our knowledge of (2).

The last result does not carry over to all arithmetical systems. We first note that it is possible to select a subset of the set of all positive integers which has the property that it contains the product of any two of its elements and also their sum. For example, we can take all positive multiples of 3. We may define a prime in this system as one which can not be expressed as the product of two elements in the system, as this is quite analogous to our definition of prime in connection with the entire set of positive integers. We note that our theorem falls down since the number 3×36 can be expressed as $(3\times4)\times(3\times1)^2$ where each factor is a prime by definition. But also

* Geometrically, the curve $u^n+v^n=1$, n odd, in the (u, v) plane has the asymptote $u=-v$ and the only rational points on it are (0, 1) and (1, 0) for the values of n for which Fermat's statement is true.

† Unique decomposition of an integer or an ideal into prime factors means unique aside from the order of the factors.

$$(3 \times 2)^2 \times (3 \times 1) = (3 \times 4) \times (3 \times 1)^2, \tag{3}$$

and each factor on the left is also prime. The two decompositions are obviously different. We can also enlarge our original system of the set of all rational integers by considering all expressions of the form $a+b\sqrt{-5}$, where a and b are rational integers. We may define a prime in this system as an element which contains no factor aside from itself and ± 1. We have

$$2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5}), \tag{4}$$

in which each factor may easily be shown to be prime according to the definition of prime, and unique decomposition does not hold in this domain.

In the discussion which follows, the main theorems, from a number of standpoints, are I, III, V, X, XI, XII, XIII, XIV. In the first place, none of the first four criteria are known to be satisfied by any value of l although a considerable number of values of l have been tested in each case. This is also true for theorems XII, XIII, and XIV when we take them together with XI. Although the theorems III and V involve x, y, and z in their statements they lead to a variety of criteria which are independent of x, y, and z for Case I of the theorem, as will be noted on reading further in this paper. By the use of theorem X we are enabled to state theorems XII, XIII, and XIV without any limitation to Case II of the theorem. *In connection with Case* I, *we shall also call particular attention to the relation* (10). *Theorems* III *and* V *are derived directly from it, and it is the principal tool employed in the proof of Theorem* X.

2. Criteria for Case I obtained by elementary methods. Legendre [33] in 1823 published the following result which he attributed to Sophie Germain.

THEOREM I. *If there exists an odd prime p such that*

$$\xi^l + \eta^l + \zeta^l \equiv 0 \pmod{p} \tag{5}$$

has no set of integral solutions ξ, η, ζ, each not divisible by p and such that l is not the residue of the lth power of any integer modulo p, then (2) *has no integral solutions each prime to l.*

We shall indicate a proof of this result. (For full proof *cf.* [12], pp. 734–735.) Set

$$\phi(y, z) = y^{l-1} - y^{l-2}z + y^{l-3}z^2 \cdots + z^{l-1},$$

then $(y+z)\phi(y, z) = -x^l$. It is known that the factors on the left are prime to each other because $x \not\equiv 0 \pmod{l}$, hence

$$y + z = a^l, \qquad z + x = b^l, \qquad x + y = c^l, \tag{6}$$

whence

$$2x = b^l + c^l - a^l, \qquad 2y = a^l + c^l - b^l, \qquad 2z = a^l + b^l - c^l. \tag{7}$$

If x, y, z are integers satisfying (2), they satisfy congruence (5), so that one of them, say x, is divisible by p. Then, by (7), $b^l+c^l+(-a)^l\equiv 0 \pmod{p}$. Consequently a, b, or c is divisible by p, employing the assumption in the theorem. It may be shown that b or $c\equiv 0 \pmod{p}$ leads to a contradiction. Hence $a\equiv 0, x\equiv 0$, $z\equiv -y, \phi(x, y)\equiv y^{l-1}, \phi(y, z)\equiv ly^{l-1} \pmod{p}$. And these give $l\gamma^l\equiv\alpha^l \pmod{p}$, for $\gamma^l=\phi(x, y)$ and $\alpha^l=\phi(y, z)$. From this the result easily follows.

Employing the theorem it is easy to see that (2) is impossible in Case I if $2l+1$ is a prime. By using this theorem Legendre [33] also proved that (2) is not satisfied for any $l<197$ in case I. (Proved for any $l<100$ by Sophie Germain.)

Dickson [8, 9] made an extended examination of the possible primes p such that (5) has no solutions in integers ξ, η, and ζ prime to p for a given l. On dividing the congruence (5) through by ζ^l, prime to p, we may write the result in the form $u+v+1\equiv 0 \pmod{p}$ where u and v are lth powers modulo p. Hence $(v+1)^m\equiv v^m\equiv 1 \pmod{p}$, if $p=1+lm$.

These congruences are also satisfied by

$$v, \quad 1/v, \quad (-1-v), \quad -1/(1+v), \quad -1-1/v, \quad -v/(1+v)$$

in place of v. They are incongruent if $2^m\not\equiv 1 \pmod{p}$. They are the roots of a reciprocal sextic equation. Set $w=v+1/v$. Then the sextic reduces to

$$c(w) = w^3 + 3w^2 + \beta w + 2\beta - 5$$

for β some function of v. This must divide $f(w)=(w+1/w)^m-1$. By proceeding along these lines Dickson found that (5) is impossible if $\xi\eta\zeta\not\equiv 0 \pmod{p}$; $p=lm+1$ for all values of $m<74$ which are not divisible by 3, as well as $m=76$ and $m=128$, except for a finite number of pairs of values l and m which pairs he was able to determine. Using these results, he [8, 9] was able to prove* that (2) is impossible in integers x, y, and z prime to l for $l<7{,}000$. As also shown by Dickson [10], (5) has integral solutions ξ, η, and ζ each prime to p for l fixed; for any prime p exceeding a certain limit. (*cf.* also Pellet, [42, 43] Hurwitz [22]).

Wendt [70] (1894) proved the following, which may also be derived from Theorem I.

THEOREM II. *If* (5) *has no solutions prime to* p, *and* $p=1+ml, m=2^v l^k, v$ *not divisible by* l, *then* (2) *has no solutions prime to* l.

3. Introduction to algebraic numbers and applications to Case I. Let us consider for a moment the problem of finding the integers prime each to each α,β, and γ such that $\alpha^2-\beta^2=\gamma^2$ is satisfied. We note immediately that the left hand member can be factored, and using the unique decomposition theorem, we infer that if integers exist satisfying this equation, then integers δ and ϵ exist such that $\alpha-\beta=\delta^2$, $\alpha+\beta=\epsilon^2$ unless both α and β are odd, in which case 2 must be inserted as a factor in each right-hand member. Now suppose we try to find a method if possible, analogous to this, for attacking the problem of finding integers x, y, and z to satisfy

* In these papers Dickson excepted the value $l=6857$, but he has stated that a proof of this case was carried out by him in manuscript.

$$x^l + y^l = -z^l.$$

As above, the left-hand member can be factored so that we have

$$(x+y)(x+\zeta y)\cdots(x+\zeta^{l-1}y) = -z^l, \tag{8}$$

where ζ is a primitive lth root of unity. But obviously we are leaving the realm of rational integers in such a factorization. However, it is possible to extend the properties of rational integers to expressions involving the lth root of unity; in short we can set up what is called the theory of cyclotomic fields, define integers in such a field and attack (2) with more generality than has so far appeared possible by means of the use of rational integers only. We introduce this topic by defining algebraic numbers in general. For expositions of the general theory *cf.* [17], [20], [21], [69e], [71].

Any subset of the set of complex numbers which contains the sum, the difference, the product and the quotient (division by zero excluded) of any two of its elements is said to form a *field*. If we have a subset of the set of complex numbers which contains the sum, the difference and the product of any two of its elements, but not the quotient in every case, the system is called a *ring*.

Let

$$f(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0,$$

where the c's are rational integers, be irreducible in the rational field (that is, $f(x) \neq g(x)h(x)$ where g and h actually contain x and all coefficients are rational) and suppose that the distinct roots of $f(x)=0$ are $\alpha_1, \alpha_2, \cdots, \alpha_n$. Then $\alpha=\alpha_1$ generates an algebraic field which is said to be of *degree n*. The other roots $\alpha_2, \alpha_3, \cdots, \alpha_n$ each generate an algebraic field and these fields are said to be the algebraic fields conjugate to that generated by α. We shall designate the field generated by α by $k(\alpha)$.

Since $f(x)$ is irreducible in the rational field, then any number θ in $k(\alpha)$ may be written uniquely in the form

$$\theta = \theta_1 = a_{n-1}\alpha_1^{n-1} + a_{n-2}\alpha_1^{n-2} + \cdots + a_0,$$

where the a's are rational. If

$$\begin{aligned}
\theta_2 &= a_{n-1}\alpha_2^{n-1} + a_{n-2}\alpha_2^{n-2} + \cdots + a_0,\\
\theta_3 &= a_{n-1}\alpha_3^{n-1} + a_{n-2}\alpha_3^{n-2} + \cdots + a_0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\theta_n &= a_{n-1}\alpha_n^{n-1} + a_{n-2}\alpha_n^{n-2} + \cdots + a_0,
\end{aligned}$$

then $\theta_2, \theta_3, \cdots, \theta_n$ are called the *conjugates* of θ_1 in $k(\alpha)$.

If $c_n=1$ then α is said to be an *algebraic integer*. The numbers in $k(\alpha)$ which are algebraic integers may be shown to form a ring, which we shall call $R(\alpha)$ or R. (In this connection we see that if $n=1$ then our definition of algebraic integer coincides with the notion of rational integer.)

Ideals. A sub-ring S of the ring R of integers in k is said to be an *ideal* in R if it is such that if θ is any integer in S then $\lambda\theta$ is in S where λ is any integer in R. It may be proved that any ideal in R may be reduced to the form

$$\lambda_1\omega_1 + \lambda_2\omega_2 + \cdots + \lambda_n\omega_n, \tag{8a}$$

where n is the degree of k, and where the ω's are fixed integers and the λ's range independently over all elements of R.

An ideal of the form $\lambda\theta$ where θ is a fixed integer in k and λ ranges over the integers in k is called a *principal ideal.* If the ideal $\mathfrak{m}$ has the form (8a) we write

$$\mathfrak{m} = (\omega_1, \omega_2, \cdots, \omega_n).$$

If $\mathfrak{a} = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $\mathfrak{b} = (\beta_1, \beta_2, \cdots, \beta_s)$ then

$$\mathfrak{ab} = (\alpha_1\beta_1, \cdots, \alpha_r\beta_1, \cdots, \alpha_1\beta_s, \cdots, \alpha_r\beta_s, \cdots)$$

is called the product of $\mathfrak{a}$ and $\mathfrak{b}$. An ideal $\mathfrak{c}$ in k is said to be *divisible* by an ideal $\mathfrak{a}$ in k when an ideal exists in k such that $\mathfrak{c} = \mathfrak{ab}$. An ideal $\mathfrak{p}$ is said to be *prime* if it has no divisor other than $\mathfrak{p}$ and the ideal consisting of all elements of R or the ideal (1). We may prove the fundamental theorem:—

Any ideal in R aside from (0) *and* (1) *has a unique decomposition into prime ideal factors.*

From this follows the remarkable result:—

Any ideal $\mathfrak{m}$ in R may be reduced to the form $\mathfrak{m} = (\alpha, \beta)$ where α and β are in R.

The last result shows that the principal ideals in R form an important subclass of R. As an example, we shall consider the ring of rational integers and any ideal in that ring. Suppose that d is the smallest integer, >0, in this ideal. Then by definition the ideal necessarily contains all integers in the form md where m ranges over all rational integers. Suppose the ideal contained an integer different from any of these, say q; by the euclidean algorithm we have $q = ad + r$, $0 < r < d$. Now q belongs to our ideal and so do a and d and consequently $q - ad$ belongs to the ideal, hence r belongs to the ideal but $r < d$. This is a contradiction, however; consequently, all the integers in the ideal have the form md and the ideal is principal. Hence all the ideals in the rational ring are principal. *However, not all ideals are principal in an algebraic field.* It may be shown that the numbers $a + b\sqrt{-5}$ with a and b rational integers are integral algebraic numbers which form a ring R. Consider the ideal $(2, 1+\sqrt{-5})$ in R. We have already noted in our discussion of the fundamental theorem of arithmetic that 2 and $1+\sqrt{-5}$ are primes. If we assume that $(2, 1+\sqrt{-5}) = (\alpha)$ it then follows that 2 is divisible by α and so is $1+\sqrt{-5}$. Since they are both primes then $\alpha = \pm 1$. Hence rational integers u, v, u_1, and v_1 exist such that

$$(1 + \sqrt{-5})(u + v\sqrt{-5}) + 2(u_1 + v_1\sqrt{-5}) = 1,$$

and this is easily shown to lead to a contradiction (*cf.* also Dickson [11], for an interesting discussion of the ideals in R). As previously noted

$$6 = 2\cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5}), \tag{4}$$

where $2, 3, 1+\sqrt{-5}, 1-\sqrt{-5}$, are all primes. However, we can now point out that the *ideal* (6) in the ring R of integers of the form $a+b\sqrt{-5}$ *has the unique decomposition into ideals*, each of which is prime,

$$(6) = (2, 1+\sqrt{-5})^2(3, 1+\sqrt{-5})(3, 1-\sqrt{-5}). \tag{9}$$

To reduce the right hand member we note, by definition of multiplication that

$$(3, 1+\sqrt{-5})(3, 1-\sqrt{-5}) = (9, 3-3\sqrt{-5}, 3+3\sqrt{-5}, 6),$$

and since $3=9-6$, the right hand member equals

$$(9, 3-3\sqrt{-5}, 3+3\sqrt{-5}, 6, 3) = (3).$$

Similarly $(2, 1+\sqrt{-5})^2=(2)$.

The relation (4) *gives two different decompositions of the number* 6 *in* R, *but* (9) *shows how we may replace this by the unique decomposition of the ideal* (6) *in* R.

We now consider in some detail the cyclotomic field (For expositions of the theory of cyclotomic fields *cf.* [3] [15], [17] I, [21].) defined by ζ where ζ is a primitive lth root of unity. Following the usual custom we do not always refer to the ring R of integers or ideals in R. We say "ideals in $k(\zeta)$" meaning ideals formed by integers in $k(\zeta)$. Any integer in this field $k(\zeta)$ can be put in the form

$$c_0 + c_1\zeta + \cdots + c_{l-2}\zeta^{l-2},$$

where the c's are rational integers. (A similar result is not true in all algebraic fields.) It would be very convenient in attacking our problem if we took (8) and supposed, in analogy with rational integers we could say that

$$x + \zeta y = (a_0 + a_1\zeta + \cdots + a_{l-2}\zeta^{l-2})^l,$$

but the situation turns out to be not nearly as simple as this. *However, in the case* $l=3$ *it is possible to show that if*

$$\lambda = 1 - \zeta,$$

then

$$x + \zeta y = \lambda\zeta^c(a_0 + a_1\zeta)^3, \qquad x + \zeta^2 y = \lambda\zeta^b(a_0' + a_1'\zeta)^3, \tag{9b}$$

$$x + y = \zeta^d\lambda^{3m-2}v^3(a_0'' + a_1''\zeta)^3,$$

where b, c, d, m, *and the* a's *are rational integers,* $m>0$, *and where* z *is divisible by* 3, (the case $xyz\not\equiv 0 \pmod 3$ can be easily disposed of without the use of relations of this type), *and it is possible to prove* (2) *impossible for* $l=3$ *proceeding from*

these relations. The examination of other values of l shows that in general the problem is more complicated. However, we find that there are products of factors in the left-hand member of (8) which take on comparatively simple forms. *This is because certain products of ideals in the cyclotomic field turn out to be principal ideals.* That is,* if $\mathfrak{p}(\zeta)$ represents an ideal in the field $k(\zeta)$ then $\prod_s \mathfrak{p}(\zeta^{s_1})$ is a principal ideal where $ss_1 \equiv 1 \pmod{l}$ and s ranges over the integers $h<l$ such that $h+|ch|<l$ where c is any integer prime to l and $|ch|$ is the least residue of ch modulo l (*cf.* Weber, [69e], pp. 748–751; Landau, [32], pp. 292–295). As examples, for $l=7$, $c=1$ we have $s=1, 2, 3$ and $s_1=1, 4, 5$ so $\mathfrak{p}(\zeta)\mathfrak{p}(\zeta^4)\mathfrak{p}(\zeta^5)$ is a principal ideal while if $l=7$, $c=2$ we have $s=1, 2, 4$ and $s_1=1, 4, 2$ showing that $\mathfrak{p}(\zeta)\mathfrak{p}(\zeta^2)\mathfrak{p}(\zeta^4)$ is a principal ideal.) Application of this gives (Kummer [29])

$$\prod_s (x+\zeta^{s_1}y) = \zeta^g(\omega(\zeta))^l, \tag{10}$$

where ω is an integer in $k(\zeta)$, and g is a rational integer. Since $(x^l-1)/(x-1)$ is irreducible in the rational field it then follows that

$$\prod_s (x+w^{s_1}y) = w^g(\omega(w))^l + A(w)\frac{(w^l-1)}{(w-1)},$$

where w is an indeterminate and $A(w)$ is a polynomial in w with integral coefficients. Setting $w=e^v$, Kummer obtains from this formula the congruences

$$B_n\left[\frac{d^{l-2n}\log(x+e^vy)}{dv^{l-2n}}\right]_{v=0} \equiv 0 \pmod{l}, \tag{11}$$

for n any integer in the set $1, 2, \cdots, (l-3)/2$ and B_n is the nth Bernoulli number defined as follows:— Consider the recursion formula $(b+1)^k=b_k$ for $k>1$. If we expand the left-hand member using the binomial theorem and substitute b_i for b^i, take $k=2$, $k=3$, *etc.* in turn, we can calculate the values of b_a, $a=1, 2, \cdots$. Thus after defining $b_0=1$, we get $b_1=-\frac{1}{2}$, $b_2=1/6$, $b_3=0$, *etc.* We then set $(-1)^{n-1}b_{2n}=B_n$. The relations (11), or (12) which follow, are known as the *Kummer criteria.* Cauchy [7, p. 362] had previously obtained the relation (10) for $c=1$ but gave no proof. He also gave without proof the result that (2) is impossible in Case I if (*cf.* Kummer [29], p. 74)

$$1+2^{l-4}+3^{l-4}+\cdots+[(l-1)/2]^{l-4} \not\equiv 0 \pmod{l}.$$

The relations (11) received little attention from mathematicians until Mirimanoff [36] in 1905 proved that they are equivalent to the conditions (if we use z and x in place of x and y, *etc.*, in (11)) given by the following theorem.

* If $\omega(\zeta)$ is an integer in $k(\zeta)$ we represent by $\omega(\zeta^a)$ the integer obtained from $\omega(\zeta)$ by substituting ζ^a for ζ. Also if $\mathfrak{a}(\zeta)$ represents an ideal in $k(\zeta)$ then $\mathfrak{a}(\zeta^a)$, $a\not\equiv 0 \pmod{l}$, represents the ideal obtained from $\mathfrak{a}(\zeta)$ by substituting ζ^a for ζ in each number of the ideal $\mathfrak{a}(\zeta)$. From this we see that the result follows that if $\mathfrak{a}(\zeta)=(\alpha_1(\zeta), \alpha_2(\zeta))$ then $\mathfrak{a}(\zeta^a)=(\alpha_1(\zeta^a), \alpha_2(\zeta^a))$.

THEOREM III. *If* (2) *is satisfied in Case* I *then*

$$B_n f_{l-2n}(t) \equiv 0 \pmod{l}; \qquad n = 1, 2, \cdots, (l-3)/2; \tag{12}$$

where if $m>1$,

$$f_m(t) = \sum_{i=1}^{l-1} i^{m-1} t^i; \quad t = -x/y, -y/x, -y/z, -z/y, -x/z, -z/x,$$

and B_n *is the nth Bernoulli number.*

The six values of t in the above are all incongruent, modulo l, except when $t=-1, \frac{1}{2}$ or 2, as the case $t^2-t+1\equiv 0$ is impossible (Pollaczek [44]). He also showed that

$$f_{l-1}(t) \equiv 0 \pmod{l}, \tag{12a}$$

and, using (12), $B_d \equiv B_{d-1} \equiv B_{d-2} \equiv B_{d-3} \equiv 0 \pmod{l}$; $d=(l-3)/2$. (The first two conditions are due to Kummer [29]). Mirimanoff [*l.c.*], also derived from (12) and (12a) the congruences

$$\begin{gathered} f_n(t) f_{l-n}(t) \equiv 0 \pmod{l}, \\ n = 1, 2, \cdots, l-1; \qquad f_1(t) = 1 + t + \cdots + t^{l-1}, \end{gathered} \tag{13}$$

which are equivalent to (12) and (12a).

In 1909 *Wieferich* [72], *employing* (12) *and* (12a), *proved that if we set*

$$\frac{m^{l-1}-1}{l} = q(m),$$

then

$$q(2) \equiv 0 \pmod{l},$$

if (2) *is satisfied in integers* x, y, *and* z *prime to* l. This remarkable result led to a series of investigations. Wieferich's argument was complicated. It was simplified by Mirimanoff [37, 38] who obtained the criterion

$$q(3) \equiv 0 \pmod{l}.$$

In 1914 Vandiver ([62b], footnote pp. 174–175) derived in a bit more complicated form the identical congruence

$$(w^l - 1)G_m^{(k)}(w) - (w^m - 1)K_m^{(k)}(w) \equiv H_m^{(k)}(w) \pmod{l}, \tag{13a}$$

where, if $b_{2s} = (-1)^{s+1}B_s$; $b_{2r+1}=0$, $r>1$; $b_1=-\frac{1}{2}$, $b_0=1$;

$$K_m^{(k)}(w) = -\sum_{n=0}^{l-2} \binom{-\frac{m}{k}}{n} b_n f_{l-n}(w),$$

$$G_m^{(k)}(w) = \frac{w(w^m-1)}{w-1} \sum_{\rho} \frac{F_1(\rho)(1-\rho^k)}{ml(w-\rho^k)},$$

$$H_m^{(k)}(w) = w(w^m - 1)\sum_{l=0}^{k-1}\sum_{\rho}\frac{\rho^l h_{l-1}^{(k,l)}(w)}{w - \rho^k},$$

$$h_n^{(k,l)}(w) = \sum_{s=0}^{p-1}(ks + l)^{n-1}w^s,$$

and where ρ is any mth root of unity, w is an indeterminate and

$$F_n(v) = \sum_{r=0}^{pm-1} r^{n-1}v^r, \quad (n > 1).$$

This relation was found by Mirimanoff [37, 38] for $k=1$ and used by him to prove $q(2)\equiv q(3)\equiv 0 \pmod{l}$. The extension to any integer k is due to Vandiver. It was this step that made the derivation of the criteria, $q(m)\equiv 0 \pmod{l}$ for $m>3$, possible. Use of the criteria (12) and (12a) with (13a) gives

$$(t^p - 1)G_m^{(k)}(t) \equiv H_m^{(k)}(t) \pmod{l}.$$

Employing this in 1914 Vandiver obtained ([52], and footnote in [62b] pp. 174–175), the criteria $q(5)\equiv q(11)\equiv 0 \pmod{l}$ and if $l\equiv 2 \pmod 3$, $q(7)\equiv q(13)\equiv 0 \pmod{l}$.

By employing a method quite different from that of Kummer, involving the law of reciprocity for lth powers, a powerful tool in the theory of algebraic numbers, we may prove

THEOREM IV. *If* (2) *is satisfied in Case* I, *then*

$$f_n(t)f_{l-n}(1 - t) \equiv 0 \pmod{l}, \tag{14}$$

$n=1, 2, \cdots, l-1$, *with* t *and* $f(t)$ *defined as in Theorem III.* (Vandiver, [59], (1925)).

Frobenius [14] (1914), Pollaczek [44] (1917), Morishima [41] (1931), and Rosser [45, 47] (1939, 1941), employing (12), (12a), (13a) and (14) proved that if (2) is satisfied in Case I then

$$q(m) \equiv 0 \pmod{l}, \tag{14a}$$

for each prime $m<44$. Using $q(2)\equiv 0 \pmod{l}$, Beeger [5] in 1939 found that this criterion does not hold for any $l<16{,}000$ except 1093 and 3511 and using Dickson's [9] results, he concluded that (2) is impossible for all primes less than 16,000. *Interpreting the results in* (14a) *by means of a new and powerful analytic method, Rosser* [46] (1941) *showed that* (2) *does not hold in Case* I *for any* $l<41{,}000{,}000$. *By extending the method of Rosser, D. H. and Emma Lehmer* [35] *in* 1941 *carried this result to* $l<253{,}747{,}889$. This is the best result to date, as far as special exponents are concerned, in Case I. In the year 1912 Furtwängler [16] proved the following.

THEOREM V. *If* (2) *is satisfied, where* x, y, *and* z *are integers prime to each other, then*

$$q(r) \equiv 0 \pmod{l}, \tag{15}$$

where r *is any factor of* x *if* x *is prime to* l *and similarly for* y *and* z. *Also the relation* (15) *holds if* r *is a factor of* $x \pm y$, *provided* $x^2 - y^2$ *is prime to* l.

It is possible to prove ([55], 78–79) this by use of the relation (10). The theorem gives easily a proof of (Vandiver [62a] (1926))

THEOREM VI. *If there exists an odd prime* p *such that* (5) *has no set of integral solutions each not divisible by* p *and such that* $p \not\equiv 1 \pmod{l^2}$ *then* (2) *has no solutions each prime to* l.

Employing the theorem of Sophie Germain together with the Theorems II and VI, Vandiver [62a] was able to prove in 1926

THEOREM VII. *If* (5) *has no set of integral solutions each prime to* p, *where* $p = 1 + ml$ *and* $m < 10\,l$, *then* (2) *has no solutions in integers each prime to* l.

If we denote by $h(l)$ the number of primes p for which (5) has no solutions with $m < 10l$ then $h(5) = 4$, $h(7) = 3$ (Dickson [8]); $h(11) = 3$, $h(13) = 4$, $h(17) = 4$, $h(19) = 6$ (N. G. W. H. Beeger, unpublished).

In 1940 Krasner [24a] proved

THEOREM VIII. *Equation* (2) *is impossible in Case* I *if there exists a prime* $p = 1 + cl$ *with* $p > 3^{c/4}$ *and* $2^c \not\equiv 1 \pmod{p}$, $c \not\equiv 0 \pmod{3}$.

Employing a result due to H. H. Mitchell and theorem VI, Vandiver [69a] proved in 1944 the following.

THEOREM IX. *If* c *is a given even integer not divisible by* 3 *and it is possible to define two primes* l *and* p *such that* $p = 1 + cl$; $l > c$ *and* $p > 3^{\phi(c)}$ *then* (2) *is impossible in integers* x, y, *and* z, *prime to* l. *Here* $\phi(c)$ *is the number of positive integers less than* c *and prime to it.*

Using (10) it may be shown that $B_s \equiv 0 \pmod{l^2}$, $s = (rl+1)/2$, $r = l-4$, $l-6$, $l-8$, $l-10$, (Vandiver [55]); and similarly for $r = l-12$, $l-14$, (Morishima [40], D. H. Lehmer [35a]) if (2) is satisfied in Case I.

If (2) is satisfied in Case I then we also have $E_{(l-3)/2} \equiv 0 \pmod{l}$ where $E_1 = 1$, $E_2 = 5$, $E_3 = 61$, *etc.* are the Euler numbers and the related criterion

$$1 + 1/2^2 + \cdots + 1/[l/6]^2 \equiv 0 \pmod{l},$$

where $[l/6]$ is the greatest integer in $l/6$ (Vandiver [61, 69b] Schwindt [48]).

By the use of (10), and some of the concepts explained in Section 4, which follows, we may prove the

THEOREM X. *If* (2) *is satisfied in integers* x, y, *and* z *prime to* l *then the second factor of the class number of* $k(\zeta)$ (defined in connection with (17) of this paper) *is divisible by* l. (Vandiver [68] (1934)).

4. Criteria governing Cases I and II of the Theorem. We shall now obtain results containing both cases of the theorem and this will enable us to give criteria for the solution of (2) without any restrictions on x, y, and z. Referring again to the relation (8) *we examine the problem: for what values of* l *may we obtain simple relations analogous to* (9b) *which we noted held in the case* $l=3$? So far, this has proved to be very difficult and the solution depends on certain concepts and results in the theory of algebraic numbers which we shall now describe. The first notion we shall consider is that of *congruence* with respect to an ideal modulus. If α and β are integers in an algebraic field k, and $\alpha-\beta$ belongs to an ideal $\mathfrak{a}(\neq(0))$, then we write $\alpha\equiv\beta \pmod{\mathfrak{a}}$. If we are dealing with rational integers $a\equiv b \pmod{m}$ means that $a-b$ is a multiple of m and this is equivalent to saying that $a-b$ belongs to the ideal (m). So this idea of congruence in an algebraic field is an extension of our ordinary idea of congruence.

The *finite number* of integers in an algebraic field k which are incongruent with respect to a given ideal $\mathfrak{m}$, is called the *norm* of $\mathfrak{m}$.

We define a *unit* in the algebraic field as an integer α such that there exists an integer α_1 in the same field with $\alpha\alpha_1=1$. Thus, ± 1 are units in the rational field, and ± 1, ζ, are units in $k(\zeta)$. The units in any algebraic field have the following properties. First, there exists a finite set, say s of them, called a *fundamental system of units*, such that any unit can be expressed as power products of these s units (that is, these units are *generators under multiplication*) multiplied by a root of unity. Also this set of units $\epsilon_1, \epsilon_2, \cdots, \epsilon_s$ is *independent*, that is from $\epsilon_1^{a_1}\epsilon_2^{a_2}\cdots\epsilon_s^{a_s}=1$ we may infer $a_1=a_2\cdots=a_s=0$. For the cyclotomic field defined by ζ we have $s=(l-3)/2$. It is known ([15], pp. 179–184, 208–209) that

$$e_i = e(\zeta^{r^i}) = \left(\frac{(1-\zeta^{r^{i+1}})(1-\zeta^{-r^{i+1}})}{(1-\zeta^{r^i})(1-\zeta^{-r^i})}\right)^{1/2} = \frac{\zeta^{r^{i+1}/2}-\zeta^{-r^{i+1}/2}}{\zeta^{r^i/2}-\zeta^{-r^i/2}} \tag{15a}$$

for $i=0, 1, \cdots, (l-5)/2$, and r a primitive root of l form a system of independent but not necessarily fundamental units. It is then easy to show ([17], II, p. 270) that integers none zero, $m_1, m_2, \cdots, m_d$; $d=(l-3)/2$, exist, such that

$$\epsilon_i^{m_i} = e_1^{c_{1i}} e_2^{c_{2i}} \cdots e_d^{c_{di}}\ ;\ i = 1, 2, \cdots, d. \tag{16}$$

We shall set

$$\begin{vmatrix} c_{11} & c_{21} \cdots c_{d1} \\ c_{12} & c_{22} \cdots c_{d2} \\ \cdots & \cdots \\ c_{1d} & c_{2d} \cdots c_{dd} \end{vmatrix} = D.$$

If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in an algebraic field k, then $\mathfrak{a}$ is said to be *equivalent* to $\mathfrak{b}$ if and only if integers α and β with $\alpha\beta \neq 0$, exist in k such that

$$(\alpha)\mathfrak{b} = (\beta)\mathfrak{a},$$

and $\mathfrak{a}$ and $\mathfrak{b}$ are said to belong to the same *class of ideals*. According to this definition, all principal ideals in k belong to the same class which will be called the *principal class*. It is obvious that all principal ideals are equivalent to the ideal (1). If $\mathfrak{a}$ is equivalent to $\mathfrak{b}$, we write

$$\mathfrak{a} \sim \mathfrak{b}.$$

As an example, we may verify that if $\mathfrak{c} = (3, 1+\sqrt{-5})$, then

$$(1 - \sqrt{-5})\mathfrak{c} = (1 + \sqrt{-5})(3, 1 - \sqrt{-5}),$$

hence

$$\mathfrak{c} \sim (3, 1 - \sqrt{-5}).$$

Now there is an infinity of ideals equivalent to $\mathfrak{c}$ since we may multiply both sides of the last equation by any principal ideal, say (δ), and the definition of equivalent shows that $\mathfrak{c} \sim (3, 1-\sqrt{-5})(\delta)$. All these ideals belong to a particular class. It is possible to show that there are just two ideal classes in the algebraic field defined by $\sqrt{-5}$ and they are the classes* to which the ideal (1) and the ideal $\mathfrak{c}$ belong.

We now state the fundamental result ([20], p. 120; [32], p. 153).

In each ideal class there is an ideal whose norm does not exceed a certain finite limit. The number of ideal classes for a given algebraic field is finite.

Since the number of classes is finite, then if $\mathfrak{a}$ is an ideal, $\mathfrak{a}, \mathfrak{a}^2, \cdots, \mathfrak{a}^k \cdots$ cannot all belong to different classes so $\mathfrak{a}^m \backsim \mathfrak{a}^n$ for some integers m and n, $m > n$.

From this it is easy to prove that

If h is the number of ideal classes in the algebraic field k, then the hth power of any ideal is a principal ideal.

In further developments we shall find that the class number h of a cyclotomic field plays a fundamental role. It is possible to find h in this particular case in finite form, in fact we find (Kummer [26])

For the class number h of the field of $k(\zeta)$, where ζ is a primitive lth root of unity, we have

* It will be illuminating to the reader for him to examine Dickson ([11], pp. 175–8) where he proves directly, without the use of the general theory, that there are exactly two ideal classes in the field defined by $\sqrt{-5}$.

$$h = \left| \frac{1}{(2l)^d} \prod_k \left[\sum_{n=0}^{l-2} r_n Z^{nk} \right] \right| \left| \frac{m_1 m_2 \cdots}{D} \right|, \tag{17}$$

where

$$d = (l-3)/2;$$

k ranges over $1, 3, 5, \cdots, l-2$; r_n *is defined by*

$$r_n \equiv r^n \pmod{l}; \quad 0 < r_n < l;$$

r is a primitive root of l; *Z is a primitive root of*

$$x^{l-1} - 1 = 0,$$

and where the m's and D are defined as in (15a) *and* (16) (*cf.* also Dirichlet-Dedekind, Zahlentheorie, 4th ed., pp. 603–634; Fueter [15] pp. 191–226). The *first factor* in h will be called h_1, the *second factor* h_2. Each is a natural number.

As an example let $l=3$; there is no fundamental unit so that we easily verify that $h=1$. For $l=5$, we find that h_1 is, since $Z=i$,

$$\left| \frac{1}{10} (1 + 2i + 4i^2 + 3i^3)(1 + 2i^3 + 4i^2 + 3i) \right| = 1.$$

We find that h_2 is unity for $l=5$, since $e(\zeta) = \zeta^2 + \zeta^3$, which is known to be a fundamental unit (Fueter [15], p. 189). The class number has been found to be unity for each $l<23$. For $l=23$ the first factor is 3, and induction would indicate that it increases rapidly from 23 onward.

Using the fact, already discussed, that the e's in (15a) form a set of independent units, we may prove the ([66], p. 749) following lemma

LEMMA 1. *If*

$$E_n = \sum_{i=0}^{(l-3)/2} e(\zeta^{r^i}) r^{-2in}, \tag{17a}$$

then

$$E_1, E_2, \cdots, E_d$$

form a set of independent units in $k(\zeta)$.

If

$$B_1, B_2, \cdots$$

are the Bernoulli numbers as in Section 3 and if

$$B_1, B_2, \cdots, B_d; \quad l > 3,$$

expressed in their lowest terms have numerators prime to l, then the prime l is said to be *regular*, otherwise *irregular*. The prime 3 is also said to be regular.

From (17) it is possible to prove, by reducing h_1 and h_2 modulo l, introducing the Bernoulli numbers and using the units in (17a),

LEMMA 2. *In order that h be prime to l it is necessary and sufficient that l be a regular prime.* (Kummer [26]).

We may also prove in a similar way

LEMMA 3. *If we have the congruence in $k(\zeta)$,*

$$\eta \equiv c \pmod{l},$$

with η a unit in $k(\zeta)$ and c a rational integer, then η is the lth power of a unit in $k(\zeta)$ provided l is regular. (Kummer [26], [27]).

An integer α in $k(\zeta)$ is said to be *primary* if an integer β exists in k such that $\alpha \equiv \beta^l \pmod{\lambda^l}$; $\lambda = (1-\zeta)$. The integer α is said to be *semi-primary* if $\alpha \equiv c \pmod{\lambda^2}$, where c is a rational integer. It is easy to show that it is possible to multiply any integer prime to λ by a power of ζ and obtain a semi-primary integer.

We shall now employ the results given above in indicating a proof of the following (Kummer [26], [27])

THEOREM XI. *If l is a given regular prime, then*

$$x^l + y^l + z^l = 0$$

is impossible for rational integers x, y, and z, none zero.

We consider the equation

$$\alpha^l + \beta^l + \gamma^l = 0, \tag{18}$$

where α, β, and γ are integers in $k(\zeta)$ such that the ideals (α), (β), and (γ) are prime each to each. We may choos α, β, and γ to be semi-primary because of the property of semi-primary integers which we mentioned in connection with their definition. If we show that this equation is impossible for l a regular prime and $\alpha\beta\gamma \neq 0$ then it will follow that (2) is also impossible for non-zero integers x, y, and z with the same restriction on l. The relation (18) gives

$$(\alpha + \beta)(\alpha + \zeta\beta) \cdots (\alpha + \zeta^{l-1}\beta) = -\gamma^l. \tag{19}$$

Now if we consider the two ideals $(\alpha+\zeta^a\beta)$ and $(\alpha+\zeta^b\beta)$ with $a \not\equiv b \pmod{l}$ and suppose them each divisible by a prime ideal $\mathfrak{p}$ then these two relations give $(\zeta^a-\zeta^b)\beta \equiv 0 \pmod{\mathfrak{p}}$. Since (β) is prime to (γ) then it follows that $\zeta^a-\zeta^b \equiv 0 \pmod{\mathfrak{p}}$ or $1-\zeta^{b-a} \equiv 0 \pmod{\mathfrak{p}}$. We divide our discussion into two cases as this congruence is not satisfied or is satisfied. In the first case we then find, since the ideals corresponding to each factor in the left hand member of (19) are all prime to each other, noting that $1-\zeta^{b-a} \not\equiv 0 \pmod{\mathfrak{p}}$ makes γ prime to λ,

$$\begin{cases} (\alpha + \beta) & = \mathfrak{a}_0^l, \\ (\alpha + \zeta\beta) & = \mathfrak{a}_1^l, \\ \cdots\cdots\cdots & \\ (\alpha + \zeta^{l-1}\beta) & = \mathfrak{a}_{l-1}^l. \end{cases} \tag{20}$$

If now h is the number of ideal classes in $k(\zeta)$ then

$$(\alpha + \zeta^a\beta)^h = (\mathfrak{a}_a^l)^h = (\mathfrak{a}_a^h)^l = (\rho_a^l)$$

since h by hypothesis is prime to l and the hth power of any ideal of $k(\zeta)$ is a principal ideal as we have already noted. Now $(\omega)=(\delta)$ gives $\omega=\epsilon\delta$ with ϵ a unit, so

$$(\alpha + \zeta^a\beta)^h = \eta_a'^l\rho_a^l,$$

where η_a' is a unit in $k(\zeta)$. And since h is prime to l there exists an integer h_1 such that $hh_1=1+lt$ and

$$(\alpha + \zeta^a\beta)^{hh_1} = \eta_a''^{lh_1}\rho_a^{lh_1},$$

then

$$(\alpha + \zeta^a\beta)^{1+lt} = \eta_a''^{lh_1}\rho_a^{lh_1},$$

and

$$\alpha + \zeta^a\beta = \eta_a''\frac{\rho_a^{h_1 l}}{(\alpha + \zeta^a\beta)^{tl}}.$$

It will then follow that*

$$\alpha + \zeta^a\beta = \eta_a\theta_a^l \tag{21}$$

where θ is an integer in $k(\zeta)$, $a=0, 1, \cdots, l-1$. It is known ([32], pp. 225–227) that $\eta_a=\zeta^a\delta$ where δ is a real unit in $k(\zeta)$. Reducing (21) with respect to the ideal $(1-\zeta)^l$ for various values of a this may be shown to be impossible for $l>3$.

For the second case, where γ is divisible by $(1-\zeta)=(\lambda)$, we assume that γ is divisible by $(\lambda)^m$ and hence we may write, if $m>1$,

$$\alpha^l + \beta^l = \mu\lambda^{lm}\omega^l \tag{22}$$

in general, where μ is a unit in $k(\zeta)$.

Proceeding by a method somewhat similar to that already used in obtaining (21) we obtain from (22)

$$\alpha + \beta = \lambda^{l(m-1)+1}\gamma_1^l,$$
$$\alpha + \zeta\beta = \nu_1\lambda\alpha_1^l,$$
$$\alpha + \zeta^2\beta = \nu_2\lambda\phi^l,$$

where ν_1 and ν_2 are units in $k(\zeta)$. Elimination of α and β from these relations gives

$$\alpha_1^l + \tau\phi^l = \tau_1\lambda^{l(m-1)}\gamma_1^l, \tag{23}$$

where τ and τ_1 are units in $k(\zeta)$. Using lemma 2 we find that $\tau=\xi^l$ where ξ is a unit in $k(\zeta)$. Set $\xi\phi=\beta_1$, in (23) we obtain

* The derivation of (21) for regular primes l is a triumph for the theory of ideals. If we had taken (19) and considered only the decomposition of γ^l into prime integers in the field $k(\zeta)$ we would at most have been able to obtain (21) for $l<23$.

$$\alpha_1^l + \beta_1^l = \tau_1 \lambda^{l(m-1)} \gamma_1^l. \tag{24}$$

This equation has the same form as (22), except the exponent of λ has been decreased by l. Proceeding with this equation as we did with (22) we obtain

$$\alpha_2^l + \beta_2^l = \tau_2 \lambda^{l(m-2)} \gamma_2^l,$$

and finally

$$\alpha_{m-1}^l + \beta_{m-1}^l = \tau_{m-1} \lambda^l \gamma_{m-1}^l,$$

with α_{m-1} and β_{m-1} semi-primary, but this is easily shown to be impossible.

The celebrated Theorem XI stands as one of the deepest results in number theory. Landau devotes the third volume of his *Vorlesungen über Zahlentheorie* to developing, in the main, the part of algebraic number theory which is needed in the proof of the theorem that (2) is impossible if l is prime to the class number of $k(\zeta)$. It would take, in the writer's opinion, at least fifty more pages, written in the style Landau employs, to prove Kummer's Theorem XI. This is mainly because of the length of the argument required to derive the closed form of the class number of $k(\zeta)$ as we gave it in (17). The argument depends on finding the class number of any algebraic field as the limit of an infinite series and showing that this limit may be given in finite form when applied to our cyclotomic field. Such able mathematicians as Dedekind, Weber, Hilbert and Fueter have given expositions of this without any of them materially simplifying the original work of Kummer.

All primes less than 619 are regular except 37, 59, 67, 101, 103, 131, 149, 157, Kummer [26, 27, 28] 233, 257, 263, 271, 283, 293, 307, 311, 347, 353, 379, 401, 409, 421, 433, 461, 463, 467, 491, 523, 541, 547, 557, 577, 587, 593, 607, 617 ([69], pp. 573–576). Jensen [23] proved that there is an infinity of irregular primes of the form $4n+3$.

We now examine equation (2) when l is an irregular prime. In all the results treated further in this paper the second factor of the class number h_2 as defined in connection with (17) occupies a central place. This number has the property (Kummer),

LEMMA 4. *The second factor h_2 of the class number of $k(\zeta)$ equals the class number of the sub-field defined by $(\zeta+\zeta^{-1})$.*

Also all the following results stem from one due to Kummer [29] (1857), which is as follows:—

The relation $x^l+y^l+z^l=0$ is impossible for rational integers x, y, and z none zero, l a given odd prime, under the following three assumptions.

1. The first factor of the class number of $k(\zeta)$ is divisible by l but not by l^2.

2. If $B_n \equiv 0 \pmod{l}$, $n<(l-1)/2$, there exists an ideal in $k(\zeta)$ with respect to which as a modulus the unit

$$E_n = \sum_{i=1}^{(l-3)/2} e(\zeta^{r^i}) r^{-2in}$$

is not congruent to the lth power of an integer in $k(\zeta)$. Here $e(\zeta^{r^i})$ is defined as in (15a).

3. The Bernoulli number B_{nl} is not divisible by l^3.

Based on the above, Kummer proved that Fermat's Last Theorem is true for $l = 37, 59$, and 67, which are irregular primes. Since the other primes less than 100 are regular, he concluded that the theorem was true for all prime exponents less than 100. We do not give Kummer's result, stated above, as a separate theorem, or discuss his argument, as it is entirely included in the following

THEOREM XII. (Vandiver [54] (1929)). *Under the following assumptions:*

(1) *the second factor of the class number of the field $k(\zeta)$ is prime to l;*

(2) *none of the Bernoulli numbers B_{nl}, $n = 1, 2, \cdots, (l-3)/2$, is divisible by l^3;*

then the equation (2) *is impossible in non-zero integers x, y, and z.*

In Case I the result follows immediately from theorem X.

In Case II the argument is a bit complicated and depends on the lemmas which follow.

LEMMA 5. *If the ideal (ω) in $k(\zeta)$ is the lth power of an ideal in that field and ω is a primary number, then (ω) is the lth power of a principal ideal in the field provided that the second factor of the class number of $k(\zeta)$ is prime to l. If the second factor of the class number is divisible by l, then at least one of the units*

$$E_n: \quad n = 1, 2, \cdots, (l-3)/2,$$

is the lth power of a unit in $k(\zeta)$. (Takagi [49a] p. 236; Vandiver [60]; [64]; [67], p. 673).

LEMMA 6. *If a unit η exists in $k(\zeta)$ which is of the form*

$$\eta \equiv c^l \pmod{\lambda^{2l}},$$

where c is a rational integer, then η is the lth power of a unit in $k(\zeta)$ except possibly when

$$B_{nl} \equiv 0 \pmod{l^3},$$

for n some integer in the set $1, 2, \cdots, (l-3)/2$. (Vandiver [54], pp. 616–621)

These lemmas were obtained by Kummer for the case where the first factor of the class number of $k(\zeta)$ is divisible by l but not by l^2. The published proofs of lemma 5 involve the use of what is called the class field theory and the law of reciprocity in the theory of algebraic numbers which concepts we cannot discuss here. The proof of Lemma 6 is direct and depends on the properties of Bernoulli numbers and the fact the E's in (17a) are independent. To handle Case II in connection with Theorem XII we write the equation in the generalized form

$$\omega^l + \theta^l = \eta\kappa^{ml}\xi^l, \tag{25}$$

where η is a unit and where (ω), (θ) and (ξ) are ideals in the field defined by $(\zeta+\zeta^{-1})$ prime each to each, $(\lambda, \xi)=1$, $m>0$. If this is proved impossible then it follows that (2) is impossible in Case II. We find, as in the proof of Theorem XI,

$$\left(\frac{\omega + \zeta^a\theta}{1-\zeta^a}\right) = \mathfrak{j}_a^l, \qquad (a = 1, 2, \cdots, l-1)$$

$$(\omega + \theta) = (\kappa)^{ml-(l-1)/2}\mathfrak{j}_0^l\,; \kappa = (1-\zeta)(1-\zeta^{-1}),$$

where $\mathfrak{j}_a$ and $\mathfrak{j}_0$ are ideals in $k(\zeta)$. Lemma 5 then gives

$$\left(\frac{\omega + \zeta^a\theta}{1-\zeta^a}\right)\left(\frac{\omega + \zeta^{-a}\theta}{1-\zeta^{-a}}\right)^{l-1} = (\rho_a')^l, \tag{26}$$

since the left-hand member is primary. But we also have since h_2 is prime to l and because of Lemma 4, if β is a real unit,

$$\left(\frac{\omega + \zeta^a\theta}{1-\zeta^a}\right)\left(\frac{\omega + \zeta^{-a}\theta}{1-\zeta^{-a}}\right) = (\rho_a'')^l\beta, \tag{27}$$

proceeding as we did in obtaining (21) from (20). Comparison of (26) and (27) gives, if η_a' is a real unit,

$$\frac{\omega + \zeta^a\theta}{1-\zeta^a} = \eta_a'\rho_a^l, \qquad (a = 1, 2, \cdots, l-1), \tag{28}$$

and since h_2 is prime to l, we may again use Lemma 4 and obtain

$$\omega + \theta = \eta_0'\kappa^{ml-(l-1)/2}\rho_0^l. \tag{28a}$$

Our proof consists of two main parts and we have finished one part with the derivation of the last relation. The next main step is to show that we can multiply ρ_a and ρ_b, the latter number obtained by substituting b for a in (28), by certain units, so that we can write

$$\frac{\omega + \zeta^a\theta}{1-\zeta^a} = \eta_a\rho_a^{*l}, \tag{28b}$$

and

$$\frac{\omega + \zeta^b\theta}{1-\zeta^b} = \eta_b\rho_b^{*l}, \tag{28c}$$

with

$$\eta_a/\eta_b = c^l \pmod{\lambda^{2l}}. \tag{28d}$$

We shall not give the steps necessary to prove the last statements. From (28d), using Lemma 6, η_a/η_b is then the lth power of a unit, using the second as-

sumption in our Theorem. In (28b) we now substitute ζ^{-1} for ζ, and multiply the two resulting relations together. This gives

$$\omega^2 + (\zeta^a + \zeta^{-a})\omega\theta + \theta^2 = \eta_a(2 - \zeta^a - \zeta^{-a})\omega^{*l}. \tag{29}$$

Also carrying out the same operation for b in place of a, where $a \not\equiv \pm b \pmod{l}$,

$$\omega^2 + (\zeta^b + \zeta^{-b})\omega\theta + \theta^2 = \eta_b(2 - \zeta^b - \zeta^{-b})\theta^{*l}. \tag{29a}$$

Then from (28a) we also have

$$\omega^2 + 2\omega\theta + \theta^2 = \eta_0\kappa^2 {}^{2ml-l+1}\rho_0^{2l}. \tag{29b}$$

The elimination of $\omega^2+\theta^2$ and $\omega\theta$ from the last three equations yields a relation which reduces, after we employ Lemma 6 with (28d), to the form

$$\omega_1^l + \theta_1^l = \delta\kappa^{(2m-1)l}\xi_1^l, \tag{30}$$

with δ a unit in $k(\zeta)$ and ω_1, θ_1, ξ_1, are integers in $k(\zeta)$. This equation has the same form as (25) with ω_1, θ_1, ξ_1, integers in $k(\zeta)$ prime to each other but $2m-1$, >0, replaces m. Comparing the value of ξ_1 with the value of ξ we see, however, that since $\xi_1=\rho_0^2$, then (ξ_1) necessarily contains a lesser number of distinct prime ideal factors than does (ξ), (aside from the exceptional case when ξ/ρ_0 is a unit in $k(\zeta)$, which is easily shown to lead to a contradiction).

Hence we obtain by repetition of the process used in connection with (25) an unlimited series of ideals in the field $k(\zeta)$, (ξ_1), (ξ_2), (ξ_3), $\cdots$, in each of which the number of distinct ideal factors is less than in the preceding, which is impossible.

The above argument was a direct extension of the argument employed by Kummer in obtaining his result given at the beginning of Section 4. Vandiver [54] in (1929) proved the

THEOREM XIII. *Under the following assumption*:

None of the units E_a, $a=a_1, a_2, \cdots, a_s$, *is congruent to the lth power of an integer in the field* $k(\zeta)$, *modulo* $\mathfrak{p}$, *where* $\mathfrak{p}$ *is a prime ideal divisor of* p, p *is a prime* $<(l^2-l)$ *of the form* 1 (mod l), *and* $a_1, a_2, \cdots, a_s$ *are the subscripts in the Bernoulli numbers in the set*

$$B_1, B_2, \cdots, B_{(l-3)/2} \tag{31}$$

which are divisible by l;

then the relation (2) *is impossible in non zero rational integers.*

As in the proof of Case II of Theorem XII the argument to prove the present theorem consists of two parts. The first part is the same as in the proof of the first part of the previous theorem (derivation of 28a) since if the first assumption in Theorem XIII holds then none of E_a's can be lth powers so h_2 is prime to l by Lemma 5. This also yields a proof for Case I by Theorem X. Proof of the second part for Case II is quite different and the method is due to Vandiver. It

consists in taking (25) and proving that ξ is divisible by a certain rational prime $p=1+cl$, and then showing that $\omega+\theta$ is divisible by p. In view of this it is possible to show that η_a/η_b in (28d) is the lth power of a unit without employing assumption 2 of Theorem XII. We can also show in (30) that $\omega_1+\theta_1$ is divisible by p and this property holds throughout our process of infinite descent which will prove Theorem XIII. In a somewhat similar way we may prove the

THEOREM XIV. (Vandiver, [54], (1929)). *Under the following assumptions:*
(1) *there exists a rational prime integer p such that the congruence*

$$u^l + v^l + w^l \equiv 0 \pmod{p}$$

has no solution u, v and w all rational integers prime to p, and $p \not\equiv 1 \pmod{l^2}$;

(2) *None of the units* E_a, $a=a_1, a_2, \cdots, a_s$, *is congruent to the lth power of an integer in the field* $k(\zeta)$ *modulo* $\mathfrak{p}$, *where* $\mathfrak{p}$ *is a prime ideal divisor of p, with p defined as in the first assumption and* $a_1, a_2, \cdots, a_s$ *are the subscripts in the Bernoulli numbers in the set* (31) *which are divisible by l*;

then the equation (2) *is impossible in rational integers none zero.*

A number of collaborators of the writer have tested the Theorems XII, XIII, and XIV for various primes l, particularly Mr. M. M. Abernathy, Dr. D. H. Lehmer, and Mr. M. E. Tittle. In the first place the assumption in Theorem XIII was tested ([67], pp. 667–670) for all irregular primes $l<619$. In each case the least prime p of the form $1+ml$ was used, and in each case it turned out that the assumption in Theorem XIII held. That is, E_a was not congruent to the lth power of an integer in $k(\zeta)$. *Hence, it was concluded* [69] [69c] *that* (2) *is impossible with x, y, and z rational integers none zero for all primes* $l<619$, since if l is regular Theorem XI provides a proof.

The fact that Theorem XIII did not break down within the limits of our computations indicates the difference in the present status of Fermat's Last Theorem from what it was as Kummer left it in his 1857 paper. One of his assumptions failed for $l=157$. If our present criteria do fail for some value of l it is unfortunate, of course, that this value did not come within the range of our calculations.

5. Discussion. I have often been asked my opinion as to the truth of the Theorem, and I should like to give here my ideas as they stand at the present time. First let us discuss Case I. I am definitely of the opinion that the theorem is true in Case I, but not because it has been verified for a great number of special cases. It is mainly because of Theorems VI and VII. Based on a great deal of experience with trinomial congruences, I think it is very likely that the number of congruences of type (5), with $p=1+ml$, $m<10l$, and those in which $p\not\equiv 1 \pmod{l^2}$ with no limitation on m, which have no solutions increase with l. Hence, in general, we could prove the first case of the theorem by the use of any one of a number of primes p. Also in the formula (13a) for $k=1$ and using (12) and (12a) we obtain

$$\frac{t(t^m-1)}{t-1}\sum_{\rho}\frac{F_1(\rho)(1-\rho)}{ml(t-\rho)} \equiv 0 \pmod{l}$$

as criteria for Case I of the theorem. The values $m=2$ and $m=3$ yield $q(2)\equiv q(3)\equiv 0 \pmod{l}$. We have left $l-3$ congruences which must also be satisfied. There is no value less than 16,000 which satisfies the first two mentioned.

In Case II I think the theorem is also true, but the indications of this are not so overwhelming. At least one of the assumptions in each of the Theorems XII to XIV fail if the second factor of the class number of the field defined by ζ, as given in (17), is divisible by l. However, about twenty-five years ago I conjectured that this number was never divisible by l. Later on, when I discovered how closely the question was related to Fermat's Last Theorem, I began to have my doubts, recalling how often conjectures concerning the theorem turned out to be incorrect. When I visited Furtwängler in Vienna in 1928, he mentioned that he had conjectured the same thing before I had brought up any such topic with him. As he had probably more experience with algebraic numbers than any mathematician of his generation, I felt a little more confident. With the second factor prime to l, I think that at least one of the Theorems XIII or XIV, if applied to the theorem for a particular l, gives a proof for l irregular with Theorem XI supplying a proof when l is regular. This is for the same reason that I mentioned in connection with certain criteria for Case I, that is, considerable experience with trinomial congruences. *However it would probably be best if I were wrong about this. I can think of nothing more interesting from the standpoint of the development of number theory, than to have it turn out that the Fermat relation has solutions, for a finite number* >0, *of primes* l.

Many mathematicians are often interested in ascertaining how a particular topic connects up with other parts of mathematics. In the case of Fermat's Last Theorem it is well known that Kummer's attempts to prove it gave rise to the theory of ideals which is now of fundamental importance in many parts of mathematics. The remarkable character of Kummer's achievement has tended, however, to minimize the great number of connections which the theorem has with other subjects. Efforts on my part to clear up the question have led me into the following topics: Bernoulli numbers and polynomials and generalizations; Euler and Genocchi numbers; Euler and Mirimanoff polynomials; partitions modulo m; finite fields and rings, including a great many types of congruences; the Dirichlet Zeta Function and the related Dedekind Function; the Lagrange resolvent and Jacobi ϕ number and various generalizations including generalized Gauss sums; theory of Kummer fields, class number, class fields, power characters and laws of reciprocity in the theory of algebraic fields; transformations of Elliptic functions and complex multiplication; Fermat's quotient and other arithmetic quotient forms; congruence theories as applied to power series; abstract algebra including, particularly, group theory and semigroups; and many types of Diophantine equations aside from the Fermat relation itself.

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THE FIRST PROFESSORSHIP OF MATHEMATICS IN THE COLONIES

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It appears that the identity of the holder of the first professorship of mathematics in what is now the United States is a moot question long since regarded as hopeless of solution by the more eminent historians in the subject. In a brochure entitled, "The Teaching and History of Mathematics in the United States," published in 1890, Florian Cajori properly gives credit to the College of William and Mary but states (page 33) that: "Owing to the repeated destruction of the college buildings and records by fire, not even the succession of the professors has been preserved . . . . The earliest professor at William and Mary whose name has come down to us was the Rev. Hugh Jones. The college had a professorship of mathematics from its very beginning, and at a date when mathematical teaching at Harvard was still in the hands of tutors only. The names of the predecessor or predecessors of Hugh Jones are not known."

The above is not a statement of fact at present although it may well have been so in 1890. President Tyler reopened the college in 1888 after a recess of seven years and at that time it is very probable that the archives, such as had survived, were in sad disorder. Now, however, due to the ceaseless researches of the librarian emeritus, Dr. Earl G. Swem, the facts are available and they portray an interesting episode in American academic history.

In casting about for a professor of mathematics the authorities very properly corresponded with the Lord Bishop of London who as chancellor of the college acted in an intimate capacity in matters of import. Upon his recommendation, according to the diary of William Byrd, the first professor of mathematics and philosophy was the Rev. Tanaquil Lefevre, who was chosen on April 25, 1711 and voted a salary of eighty pounds a year. He was a man of excellent education and background. His father, who spelled his name Tanaguy Lefebre, was for thirty years a French minister to Switzerland and England. Furthermore the records seem to indicate that he had a competent knowledge of mathematics and in 1714 published a work on algebra jointly with his son. Under the circumstances it was to be expected that the new addition to the college faculty would have brought scholarship and dignity as the handmaidens of success. Apparently, however, his interests were not wholly mathematical in view of the fact that on May 8, 1712 we find the authorities addressing the following information to the honorable Lord Bishop:

"I gave your lordship an account of Mr. Lefevre's admission into the college upon your lordship's recommendation and aim to acquaint you now that after a tryal of three quarters of a year he appeared so negligent in all of the posts of duty and guilty of some other very great irregularities, that the governors of the college could no longer bear with him, and were obliged to remove him from office, though at the same time out of regard to the honorable recom-

mendations he brought with him they continued his salary for four months longer than he officiated. I am apt to believe that most of his irregularities were owing to an idle hussy he brought over with him, because since she left him (I got her a passage back to England last February) he has left off that scandalous custom of drinking and appears quite another man, being now settled at a gentleman's house for teaching his son, and has a competent salary enough to keep him from being any more burdensome to your lordship or his other friends . . . ."

Just how long Lefevre remained in the colonies is not known but that his dismissal caused him both pecuniary and mental grief is evidenced by a petition he submits on August 17, 1735 in which through the medium of rather ponderous and obscure rhetoric his embarrassment is obvious:

"I humbly recommend myself, since I have got a sham employment who hath ruined me, let by the laws of the Universal God a distressed clergyman be relieved by a more successful warrant from the Society Justice and Charity are Petitioners that I might be indemnified so that I be able to have bread till I be sent by the Society to a real business about Divinity, Philosophy, Mathematics or Grammar, either in Greek or Latin,

"I will be sufferer if I do not find again ye included-papers when I pleased a Society for ye propagation of the faith is altogether for the Propagation of Charity. This is the only foundation of the humble petition presented to you.

"By your most humble Svt. Tanaguy Le Fevre
Tanaquillus Febre

"The employment of head master of Wm and Mary College hath been given several years ago to one Blackmore by ye bare hog drivers Authority & Power."

Successive perusals of the above cause increasing query as to whether the writer had fully relinquished some of his weaknesses of former years. In any event the document stands for whatever it may be worth.

The foregoing details are particularly eloquent when taken together with the fact that the first Hollis professor of mathematics at Harvard, Isaac Greenwood, who took office in 1727, was finally haled before the exasperated officials, to whom by 1738 forbearance had ceased to be a virtue, and likewise expelled for causes including intemperance. It took Greenwood longer in the more austere New England climate to liquidate himself but in any event the evidence is conclusive that mathematics got off to a rollicking, if not wholly dignified, start in the colonies. Those who today assert that of yore the teachers of the queen of the sciences lacked the human touch are grossly unaware of what might be termed the initial conditions.

After the Lefevre debacle the authorities moved with great deliberation, so much so in fact that it was not until 1717 that the Rev. Hugh Jones mentioned by Cajori was appointed. He was a Cambridge graduate of outstanding culture and scholarship. His monograph on the "Present State of Virginia" is acknowledged to be one of the best sources of information respecting the colony in the early part of the eighteenth century.

In 1729 there were six professors, all graduates of Edinburgh, Oxford or Cambridge.

By its charter the college was empowered to grant licenses to the county land surveyors in return for which the recipient was to pay one sixth of his fees received. It is a point of interest that in 1749 such a license was granted to George Washington. Such an appointment was at that time the equivalent of a degree in civil engineering. He was doubtless examined in part by John Graeme then professor of mathematics.

In 1770 the professor of natural philosophy also taught "physicks, metaphysics and mathematicks" at a salary of eighty pounds a year plus twenty shillings for each scholar. In addition each professor was the incumbent of a neighboring church and received therefrom an additional consideration of sixteen thousand pounds of tobacco.

It would appear that as time went on there arose an alarming demand for permission to take only courses in mathematics. In consequence we find the professors framing the following opinion as of May 2, 1770:

"With respect to the proposal of the visitors and governors for setting all such youths, whether resident in or out of the college, who have acquired a competent knowledge of common or vulgar arithmetic, and whose parents or guardians may desire it, to be received into the mathematical school. The president and masters or professors beg leave to represent that the college is not designed to be the sole place of resort for education in the colony, but the best place for training up youth who are intended to be qualified for any of the three learned professions, and accomplished citizens, in a regular course of study.

That the plan or method for this regular progress in study hitherto approved in the most famous universities as well as in the Statute of William and Mary College consists in the pursuit first of classical knowledge 2ndly of philosophy natural and moral and lastly of such sciences as are to become the business of the students during the remainder of their lives.

That the above method cannot be departed from or occasionally altered even for the sake of extraordinary geniuses who are able to shine in a learned profession without the foundation of classical learning, much less can it be accomodated to the views of those who aim at no more than a skill in Vulgare Arithmetic and some practical branch of mathematics to qualify them for an inferior office in life, without doing abundantly more injury than benefit to the public.

. . . . . .

That the number of those whom we call irregulars from our unwillingness to use a more adequate term, may soon grow great enough to employ the whole time of the professor of N.P. and Math. in teaching a practical branch or two of the mathematics, and all such irregular and desultary students after quitting the college, will be very apt to seize every opportunity to boast of their University education, under which disadvantage alone we think it almost impossible that the college should ever rise to any very high place of renoun."

However narrow the foregoing may appear as viewed from the present it nevertheless possesses a pith and lucidity that commands admiration. One can imagine that the document stirred up somewhat of an academic tempest concern ing matters curricular in view of the fact that in the local paper, the Virginia Gazette, under date of June 9, 1774 one signing himself "A.B." comments as follows: "I shall not ask why algebra is to precede the first elements of geometry, why fluxions must precede spherical trigonometry, or conic sections: it would be as useless as absurd. The method which has been for three or four years adopted as such has received the sanction of a Barrow, a Newton, a McLauren, a Whiston, and a Sanderson."

A complete list of the professors of mathematics up to the time of the closing of the college by the Civil War is as follows:

Tanaquil Lefevre	1711	Thomas Gwatkin	1769
Hugh Jones	1717	James Madison	1773
Alexander Irvine	1728	Robert Andrews	1784
Joshua Fry	1732	George Blackburn	1805
John Graeme	1737	Ferdinand Campbell	1811
Richard Graham	1749	Robert Saunders	1833
William Small	1758	Benjamin S. Ewell	1849
Richard Graham	1763	Robert Gatewood	1854
John Camm	1766	James M. Wise	1855
		Thomas T. L. Snead	1856

Robert Andrews was probably the first to teach mathematics only.

RETROSPECT

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1. These five years. This is the last issue of the MONTHLY to appear under my editorship. As I pause and look back over the past five years and while the details are fresh in my mind, it seems appropriate to relate for the readers something of the story of this eventful period.

My editorship has spanned the war years. The Japanese attack on Pearl Harbor occurred while the first issue was being set into type. We are reading proof on the last number while treaties of peace are being debated in a broken world. The four dozen issues between them were produced amidst all sorts of difficulties engendered by the war. If the editor who took up his duties so lightheartedly five years ago could have foreseen the future, would he have recoiled from the task? I am not sure.

2. Secretarial troubles. The most serious single difficulty arose from the scarcity of secretarial help. Chicago developed rapidly as a center of war industry and hundreds of office staffs were hastily built up. Any young woman with a

slight acquaintance with a typewriter could get a position at an inflated salary. One after another the secretaries left us after a stay of a few weeks each. I have no idea how many there were and their very names are forgotten.

It takes a secretary several months to learn the details connected with the editorship of the MONTHLY. For nearly three years the work was done by girls who never got beyond the first few lessons in the duties of the position. There came finally one dark time when, for weeks, it seemed that another secretary was nowhere to be found. The editor seriously considered employing a mathematician to teach his classes—mathematicians being somewhat plentiful at the moment—and serving as his own typist. It never quite came to that desperate improvisation. The best of all secretaries appeared magically upon the scene. She remains to this day.

3. Printing difficulties. I want to explain to our readers that for years it was never possible to get the MONTHLY out on time. This was never anybody's fault: it was inherent in the war effort. We had to stand aside, of course, while work for the armed services was rushed through. From first to last the George Banta Publishing Company printed ten million books for the Army and Navy.

The following are comments of a member of the Banta staff.* "Before the echoes of the Pearl Harbor catastrophe had died down, we were beset with a series of countless headaches which are concomitant with any such abrupt transition. But these headaches had to be endured with patience, and the new work proceed as best and with what speed it might, for the word had gone out from Washington and from Annapolis that this or that book, in such and such quantities, had to be put on our presses—and *now*!"

"Production plans and schedules were impossible of maintenance; a working schedule planned in the morning would be disrupted and changed through a directive or a priority plea by noon. And so it went; customers given a promise made in good faith were at times sadly disappointed—their work had to be delayed when war requirements demanded."

During the latter part of the war the shortage of skilled workmen made itself felt. The young printers, like the young of all callings, went into the service. The older men carried on. Some of our best proof came from the hands of these experienced men, but they had too much to do. There were delays and more delays.

The amount of paper available to us was reduced by governmental regulation. Consequently, it was necessary to reduce the number of pages printed during the year. Also, it was required that we adopt a cheaper grade of paper than we had used formerly. The result was a somewhat less attractive product. Photographic reproduction, for example, became impossible. These were small sacrifices.

The censorship was never a burden. Mathematical articles are seldom of a sort to give either aid or comfort to the enemy. Our readers had to forego a couple of papers on cryptography and one or two other small things.

* A. J. Hyson in the Banta shop newspaper, *Nightmare*, September-October, 1945.

4. Staples and the war. The most amusing of all the war regulations was never put into effect. The pages from which you are reading are held together by two wire staples. It occurred to some planner that half of this metal might be saved for the making of munitions and a regulation was promulgated that only one staple should be used. Many of us felt that the havoc done to our magazines could not possibly be justified by the gain in war materials, which must indeed have been infinitesimal. The order was withdrawn.

5. The stream of papers. The mathematicians, like others, put their shoulders to the wheel. The younger men went into the service. Many of all ages entered technical fields in connection with the armed forces or in war industries. Oldsters continued to teach, working with V-12 or ASTP, devoting long hours to the hurried instruction of heterogeneous groups, foregoing the usual summer vacations. Some followed all this with a period of teaching at Biarritz, Shrivenham, or Florence.

It is difficult to see who had time to write mathematical papers. Yet the stream of articles continued to flow. In all, some 450 longer papers passed through the hands of the editor-in-chief. There was never a time when there was a lack of material for the printer and there was never a conscious sacrifice of standards.

Some papers came, strangely enough, from abroad. There were some from India and several from the interior of China. France and Spain were represented by the indefatigable M. Thébault. We printed articles from Great Britain, Belgium, Australia, Brazil, and Puerto Rico. But most of the papers came from the United States and Canada.

6. Rejected papers. After careful study by competent referees about forty per cent of the papers presented were adjudged unsuitable for publication in the MONTHLY. Why? Perhaps an attempt to analyze the reasons for rejection will be of some help to future authors. Certainly a higher rate of acceptability will lighten the tasks of my successor.

There were a few papers which had little to do with mathematics and there were some which were simply wrong. But the main causes of rejection were three in number.

(1) Some papers are too technical for our readers. They are sound mathematically but they lie in fields of advanced research and narrow specialization. They would be read by few in the collegiate field. A paper which lacks some universality of appeal probably should be sent to a research journal.

(2) Some papers are what might, in not too derogatory a sense, be called trivial. They develop at length results that might be considered as mathematical exercises. They might better form contributions to the problem departments or to Discussions and Notes.

(3) Very often papers are poorly written. Fifty per cent of the papers which are ultimately accepted have to be revised. Nobody who has not been an editor can realize how much bad writing college professors can do. Many a paper composed in a lengthy, prolix, and awkward prose might have been saved by a

sprightly and fluent style. Those who write on the teaching of mathematics are still the chief sinners in this group. My advice to you would be to swallow your pride and show your paper to a friend in the English department.

7. Angle trisections and Fermat's theorem. I have not included in the previous count the long list of angle trisections and proofs of Fermat's last theorem. The diverting correspondence connected with these has done much to lighten the editorial load. The authors are blest with a boundless confidence and enthusiasm. "The new discovery is so important," says one trisector, "that I am in hopes of being invited to attend the Annual Meeting of the Association." Much of their work is copyrighted.

Sometimes the writer seeks profit, but this is exceptional. "For the disclosure of my discovery I should like to have some financial benefit" runs a letter from the West Indies. Another hopes that our publication of his results will help him "to find some one to finance the construction of my machine for interplanetary navigation."

My letters telling the authors firmly but not too bluntly that their work is erroneous have elicited varied responses. Some are angry. "I have become the dupe of bad incompetency." "I am very suspicious of your qualifications as to judging fairly the merits and demerits of my paper." Usually, however, the news is received in a spirit of resignation, born, no doubt, of oft-repeated experience. "I cannot make this any plainer and shall leave the matter in the hands of the university experts." "You know what happened to the men who first proclaimed that the earth is round." Usually we have parted friends, but I am sure that I have never convinced anybody of his errors.

This issue of the MONTHLY contains a scholarly article on Fermat's last theorem. I have no doubt that its author will received many "proofs" from these old friends of mine.

8. The associate editors. The publication of the MONTHLY has been a cooperative undertaking. Foremost among the collaborators have been the dozen and a half associate editors. Those who have been in charge of departments have met and solved their own considerable editorial problems. All have given unstintingly of their time year after year to the end that the magazine might eventually reach your desk. My most pleasant recollections during these five years are of their unselfish devotion.

One of these will now become editor-in-chief for the period 1947–1951. Professor C. V. Newsom, editor during the war of the widely read department of War Information, brings to his task both editorial experience and familiarity with the affairs of the Association. He will be spared many of the difficulties of the war years and he can devote his full thought to the building of a greater MONTHLY. May he achieve a matchless success!

DISCUSSIONS AND NOTES

EDITED BY E. F. BECKENBACH, University of California, Los Angeles 24, Calif.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

AN INTEGRAL

C. I. LUBIN, University of Cincinnati

It is pointed out early in the study of integrals in a course in advanced calculus that an integral of an integrable function is a continuous function of the upper limit.

Consider the integral,

$$F(x) = \int_{-1}^{x} \frac{e^{1/y}}{y^2(e^{1/y}+1)^2}\,dy. \tag{1}$$

The integrand,

$$f(y) = \frac{e^{1/y}}{y^2(e^{1/y}+1)^2}, \qquad (y \neq 0),$$

$$f(0) = 0,$$

is continuous in the interval $-1 \leqq y \leqq k$, k some positive number, and so the function $F(x)$ must be continuous in the same interval.

Now

$$\frac{d}{dx}\left(\frac{1}{e^{1/x}+1}\right) = f(x), \qquad (x \neq 0),$$

so that at first glance one might be tempted to write the integral (1) as

$$\frac{1}{e^{1/x}+1} - \frac{e}{e+1}. \tag{2}$$

The function (2), however, is discontinuous at $x=0$. This means that the continuous function $F(x)$, for the interval $-1 \leqq x \leqq k$, cannot be represented by (2).

The appropriate representation is as follows:

$$F(x) = \frac{1}{e^{1/x}+1} - \frac{e}{e+1}, \qquad (-1 \leqq x < 0),$$

$$F(0) = \frac{1}{e+1},$$

$$F(x) = \frac{1}{e^{1/x}+1} + \frac{1}{e+1}, \qquad (0 < x \leqq k).$$

A WELL-KNOWN INTEGRAL

H. F. SANDHAM, Trinity College, Ireland

If we let

$$\theta = \int_0^\infty e^{-x^2} dx,$$

then we have

$$\theta = \int_0^\infty e^{-x^2y^2} y\,dx,$$

or

$$2e^{-y^2}\theta = \int_0^\infty e^{-y^2(1+x^2)} 2y\,dx.$$

Integrating from 0 to ∞ with respect to y, and interchanging the order of integration, we get

$$2\theta^2 = \int_0^\infty \frac{dx}{1+x^2} = \tan^{-1} x \Big]_0^\infty = \frac{\pi}{2},$$

whence

$$\int_0^\infty e^{-x^2} dx = \tfrac{1}{2}\sqrt{\pi}.$$

A REMARK ON THE THEORY OF PARTITIONS

RAYMOND REDHEFFER, Massachusetts Institute of Technology

1. The partition function $p(m, r, n)$ represents the number of ways of distributing n indistinguishable objects in r indistinguishable boxes without putting more than m objects in any box. In this note we use very elementary methods to establish that

$$\sum_{n=0}^{mr} p(m, r, n) = \frac{(m+r)!}{m!r!} \tag{1}$$

for all positive integers m, r, both sides of the equation being defined to be unity when $mr = 0$.

2. *Proof.* Denoting the sum in (1) by $A(m, r)$, we make all possible distributions of 0, 1, 2, $\cdots$ objects in the r boxes, without putting more than $m-1$ objects in any box. The number of distributions of this type is $A(m-1, r)$, by definition. Next, let us put m objects in some one box and make all possible distributions of 0, 1, 2, $\cdots$ objects in the remaining $r-1$ boxes, this time allowing as many as m objects in any box. The number of different arrangements is now $A(m, r-1)$. We shall show that all arrangements of the first type, plus all of the second type, will give the total number of arrangements $A(m, r)$, so that

$$A(m, r) = A(m-1, r) + A(m, r-1). \tag{2}$$

a) All arrangements are included. For the first type includes all in which no box has more than $m-1$ objects, while the second includes all in which at least one box has m objects.

b) The same arrangement is not counted twice. For no arrangement of the first type is equal to an arrangement of the second type, since in the first case no box has more than $m-1$ objects, while in the second case one box always has m objects. Moreover, the arrangements of the first type are all different from each other, by the definition of $A(m, r)$, and the same is true of the arrangements of the second type.

Having established (2), let us suppose that (1) holds for $r+m=k$. Then it must hold for $r+m=k+1$; for (2) is simply the well-known law of formation for the binomial coefficients, as used in constructing Pascal's triangle. Equation (1) obviously holds for $r+m=0$ or 1, and thus the induction is complete.

3. The result may also be proved from the well-known fact that successive summation, starting with 1, will lead to the binomial coefficient; or, alternately, this latter result can be established by means of (1).

ON A CONJECTURE OF ERDÖS

F. BAGEMIHL, University of Rochester

In a paper appearing in this MONTHLY, vol. 53, 1946, p. 248, P. Erdös made the conjecture that on every convex curve there exists a point P such that every circle with center P intersects the curve in at most two points. We shall show that this conjecture is false by taking as our convex curve an equilateral triangle. If we describe about a vertex of the triangle a circle of radius greater than an altitude but less than a side, about the midpoint of a side of the triangle a circle of radius equal to one half a side, about any other point of the triangle a circle of radius equal to the maximum of the distances of the point from the two adjacent vertices, then in each case the circle in question will intersect the triangle in more than two points.

It is obvious, by continuity, that we can obtain from our triangle a convex curve containing no rectilinear arc and not having the conjectured property.

QUATERNIONS AND REFLECTIONS (POSTSCRIPT)

H. S. M. COXETER, University of Toronto

Theorem 5.1* was given five years earlier (with an even simpler proof) by Ernst Witt, Spiegelungsgruppen und Aufzählung halbeinfacher Liescher Ringe, Abhandlungen aus dem Mathematischen Seminar der Hansischen Universität, vol. 14, 1941, p. 308.

* This MONTHLY, vol. 53, 1946, p. 141.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to Loyal F. Ollmann, Hofstra College, Hempstead, N. Y.

CLUB REPORTS 1945-46

Mathematics Club, University of Dayton

Biweekly meetings were held during the year at which members of the club presented the following papers:

Heuristic reasoning, by Catherine Ens

Women in mathematics, by Virginia Johnston

Points of intersection of concentric circles, by Alice Blaeser

Unsolved riddles in mathematics, by George Falkenbach

Mathematics and navigation, by Robert Huels

The Bernoulli family, by Margaret Magin

Diophantine analysis, by Alice Blaeser

Mathematics and genetics, by Thomas McCarthy.

Faculty talks presented during the year were:

Are irrational numbers rational? by Dr. K. C. Schraut

A consideration of the wine glass problem of calculus, by Dr. Schraut

The statistical analysis of tests, by Professor C. G. Peckham

What is an infinite series?, by Dr. Otto Szasz of the University of Cincinnati.

A picnic and wiener roast was held in a Dayton City Park during the first semester, and an annual banquet at the Wishing Well Inn during the second semester.

At the formal initiation ceremony, King Bradow delivered the Vice President's charge to the five new members on the theme

The Mathematics Club member, his privileges and responsibilities.

The *Dean of Science Awards* were conferred each semester for the most interesting student paper. The recipients, George Falkenbach and Thomas McCarthy, received copies of Men of Mathematics by E. T. Bell. The *Mathematics Club Alumni Awards of Excellence in Advanced Mathematics* were conferred at Commencement upon a senior, Catherine Ens, and a junior, George Falkenbach.

The sum of $100.00 was contributed by the Mathematics Club Alumni Fund to the Albert Emmanuel Library for the purchase of books, and a suitable book plate was designed for these books.

The year was marked by progress in the National Mathematics Honor Society of Secondary Schools which was founded under the auspices of the Mathematics Club. Club officers participated in Chapter installations in Our Lady of Angels High School in St. Bernard and Our Mother of Mercy High School in Cincinnati.

The officers of the Club during the year 1945–46 were: President, Catherine Ens; Vice-President and Secretary, King Bradow; Treasurer, James Noll. Dr. K. C. Schraut was Faculty Adviser.

Mathematical Society, University of Adelaide

Apart from a preliminary meeting, on March 22, thirteen regular meetings were held during the academic year 1945 when the following papers were presented:

Fields of complex numbers with real coordinates, by M. C. Gray.

Rings with divisors of zero, by G. E. Wall.

Groups and ornaments, by Mrs. M. Sved.

An application of the method of least squares, by E. A. Cornish.

The linear oscillator, a boundary and eigen-value problem, by W. O. Gibberd.

An eigen-value problem of quantum mechanics, by H. Schwerdtfeger.

In how many ways can a 2-shilling piece be changed into smaller coins?, by Mrs. H. Schwerdtfeger.

Constructions with ruler and compass, by G. E. Wall.

The irrationality of pi, by M. C. Gray.

Some problems of particle motion in viscous fluids, by W. H. Schneider.

On transcendental numbers, by Mrs. M. Sved.

A distribution problem in mathematical statistics, by A. T. James.

In order to meet the interests of students of Mathematical Statistics and to relieve some third- and fourth-year courses, the *Mathematical Society*, during the year 1946, will be conducted as a *Colloquium on the theory of the gamma-function and its applications.*

Kappa Mu Epsilon, Mount St. Scholastica College

The Kansas Gamma Chapter numbered 19 active members and 13 active pledges. Meetings were held semi-monthly. The theme for the year's activities was

Various types of mathematical presentation: to the general public, to the practical mathematician, to the scientist, and to the theoretical mathematician.

Mathematics and the college student, by the Reverend Pius Pretz, O.S.B., Professor of mathematics at St. Benedict's College, Atchison, Kansas, was the lecture given at an open meeting.

The chapter continued the quarterly publication of the *Exponent*, the chapter paper.

Following the initiation in May of nine new members, a buffet supper was served. It was announced that Mary Lou Maloney received the annual award for having contributed the most to the fraternity during the year.

Officers for the year 1946–1947 are: President, Mary Jane Fox; Vice-President, Joanne Shuey; Secretary, Victoria Fritton; Treasurer, Jean Moran; Chapter Publicity, Frances Knightley; Chapter Musician, Elizabeth Gulde; Corresponding Secretary and Faculty Sponsor, Sister Helen Sullivan, O.S.B.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND HOWARD EVES

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 746. *Proposed by H. F. Sandham, Trinity College, Ireland*

Show that for all positive integers r

$$r\{1-(n+1)^{-1/r}\} < 1/1 + 1/2 + \cdots + 1/n \leqq r(n^{1/r}-1)+1.$$

E 747. *Proposed by H. W. Becker, Omaha, Nebraska*

Can we decompose an integer force F into n integer forces $F_1, \cdots, F_n$, such that the sum of any number of the components is also an integer force, where $n>2$?

E 748. *Proposed by George Pólya, Stanford University*

Let α be the angle between the axis and any element of a right circular cone. If the cone is cut through by a plane (not necessarily perpendicular to the axis) show that the lateral area of the part remaining between the plane and the vertex is given by $\pi AG \sin\alpha$, where A and G are respectively the arithmetic and the geometric mean of the longest and the shortest remaining elements.

E 749. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a given sphere inscribe a right circular cone whose lateral area is equal to the area of the zone beneath its base. Show that the total area of the cone is equal to the area of the zone in which it is inscribed.

E 750. *Proposed by Paul Erdös, Stanford University*

Find the number of intersections of the diagonals of a convex polygon of n sides.

SOLUTIONS

Iterated Factorials

E 684 [1945, 395 and 1946, 271]. *Proposed by Paul Erdös, Stanford University*

Prove that $n!^{(n-1)!}$ divides $(n!)!$; and that if n is not a power of a prime, $n!^{n(n-2)!}$ divides $(n!)!$.

Note by Frita Herzog, Michigan State College. I believe that the solution as given (*cf.* [1946, 271]) is in error, as far as the proof of the second part of the problem is concerned. The statement that "since n is not a power of a prime, the number of times that p divides $n!$ is the same as the number of times that p divides $(n-1)!$" holds true if and only if p does not divide n. Thus, in order to complete the proof, the case in which p is a prime divisor of n ought to be considered.

Let p^a be the highest power of p that divides n and let the integer b be defined by $p^b<n<p^{b+1}$. (Here the fact is used that n is not a power of a prime.) Obviously, $1\leqq a\leqq b$. We have to show that

$$\sum_{i=1}^{\infty}\{[n!/p^i]-n(n-2)![n/p^i]\}\geqq 0. \tag{1}$$

The terms of the series in (1), corresponding to $i=1, 2, \cdots, a$, are actually negative, namely, since n/p^i and $n!/p^i$ are integers, equal to $-n(n-2)!/p^i$. Thus their sum equals

$$-n(n-2)!\sum_{i=1}^{a}1/p^i.$$

To show that this negative number is offset by a positive number of equal or greater absolute value, I remark first that for $i>a$, p^i does not divide n, hence $[n/p^i]=[(n-1)/p^i]$ and the method employed in the published solution shows that no terms with $i>a$ of the series in (1) can be negative. Moreover, consider the terms of that series, corresponding to $i=b+1, b+2, \cdots, b+a$, We have for $j=1, 2, \cdots, a$, by the use of $p^b\leqq n-1$,

$$n!/p^{b+j}=\{n(n-2)!/p^j\}\{(n-1)/p^b\}\geqq n(n-2)!/p^j.$$

The last number being an integer, we obtain $[n!/p^{b+j}]\geqq n(n-2)!/p^j$ for $j=1, 2, \cdots, a$. On the other hand, $p^{b+j}>n$ and hence $[n/p^{b+j}]=0$. Thus the sum of the terms in (1), corresponding to $i=b+1, b+2, \cdots, b+a$, is greater than or equal to

$$+n(n-2)!\sum_{i=1}^{a}i/p^i.$$

I also remark that, *when n is the power of a prime, then $n!^{n(n-2)!}$ does not divide $(n!)!$*; in other words, the condition of the second past of the problem is necessary as well as sufficient,

Let $n=p^a$. It suffices to show (see (1)) that

$$\sum_{i=1}^{\infty}\{[p^a!/p^i]-p^a(p^a-2)![p^a/p^i]\}<0. \tag{2}$$

The sum of the first a terms in (2) is again, as above, equal to

$$-p^a(p^a-2)!\sum_{i=1}^{a}1/p^i=-(p^a-1)!/(p-1).$$

For $i>a$, we have $[p^a/p^i]=0$ and $[p^a!/p^i]\leqq p^a!/p^i$, but actually $<$ for sufficiently large i, Hence the sum of the remaining terms of the series in (2), beginning with $i=a+1$, is less than

$$\sum_{i=a+1}^{\infty} p^a!/p^i = (p^a - 1)!/(p - 1).$$

This completes the proof of (2).

Sir William Hamilton's Icosian Game

E 711 [1946, 156]. *Proposed by H. S. M. Coxeter, University of Toronto*

Suppose that the vertices of a polyhedron represent places that we wish to visit, while the edges represent the only possible routes. Hamilton considered the problem of visiting all the places, without repetition, on a single journey. (See, *e.g.*, W. W. R. Ball, *Mathematical Recreations and Essays*, London, 1939, p. 262.) This is easily solved for the pentagonal dodecahedron. Prove that it cannot be done for the rhombic dodecahedron.

Solution by Arthur Rosenthal, University of New Mexico. The rhombic dodecahedron has eight three-edged vertices T and six four-edged vertices F. The neighbors of each T are F's and neighbors of each F are T's. Hence for each journey the T's and F's have to alternate. But eight T's and six F's cannot be written as one alternating sequence, whether we wish to return to the starting point or not.

Also solved by Norman Anning, J. B. Kelly, N. D. Lane, Leo Moser, and E. D. Schell.

A Set of Three Numbers

E 718 [1946, 219]. *Proposed by Roy Dubisch, Syracuse University*

Let A be a positive integer of r digits given by $A=\sum_{i=1}^{r} x_i 10^{i-1}$ and define $D(A)=(\sum_{i=1}^{r} x_i)^2$. By $D^n(A)$, where n is a positive integer, we mean the result of applying the operator, D, n successive times to A. Prove that for every A there exists an n such that $D^n(A)=1$, 81, or 169. (*Cf.* Porges, *A set of eight numbers*, this MONTHLY [1945, 379–384]).

Solution by Paul Bateman, Philadelphia, Pa. It is easily seen that for $A\geqq 400$, $D(A)<A$. Hence, if $A\geqq 400$, there exists an n such that $D^n(A)<400$. Hence it suffices to prove the assertion for $A<400$. If $A<400$, then clearly $D(A)\leqq(3+9+9)^2=21^2$ and $D(A)$ is one of the squares $1^2, 2^2, \cdots, 21^2$. Verification of the theorem for these twenty-one numbers completes the proof.

Also solved by Paul Brock, R. C. Buck, Daniel Finkel, Free Jamison, W. J. LeVeque, E. P. Starke, and the proposer

Starke also showed that if A has four or less digits, then $n\leqq 4$. He also showed that the end result is 1 whenever $A\pm 1$ is divisible by 9, 81 when A is divisible by 3, and 169 otherwise.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, N. J. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extension of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS AND SOLUTION

4225. *Proposed by H. F. Sandham, Trinity College, Ireland*

There are given N points in a plane, the roots of $f(z)=0$. A conic is constructed with center at the mean center, and is such that the sum of the squares of the perpendiculars from the points on to a line through the center is $N(N-1)$ times the square of the perpendicular from the center on to a parallel tangent. Prove that the foci are given by $f^{(N-2)}(z)=0$.

4226. *Proposed by Paul Erdös, University of Michigan*

Let $n!=\Pi_{p\leq n}p^{k_p}$ (p, primes). Assume that $p_1>p_2$ and $k_{p_1}<k_{p_2}$. Prove that $p_1^{k_{p_1}}<p_2^{k_{p_2}}$.

4227. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In what number systems is the square of a number of three digits of the form *bcb* of the form *aabbcc*?

4228. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a tetrahedron $ABCD$ the straight lines joining each vertex to the points of intersection V and W of the six spheres of similitude of four given spheres with centers A, B, C, D, meet the circumsphere in the vertices of two equal tetrahedrons $A'B'C'D'$ and $A''B''C''D''$.

SOLUTIONS

Probability of an Uncrowded Road

4161 [1945, 345]. *Proposed by R. E. Gaines, University of Richmond*

Along a straight road a miles long are n persons. What is the probability that no two persons are less than b miles apart?

Solution by A. G. Clark, Colorado A & M College.

With one end of the road as an origin, let the position of any one person be represented by x, $0\leq x\leq a$. Then x is a random variable with a distribution law:

$$p(x)=\begin{cases}1/a; & 0\leq x\leq a\\ 0; & \text{elsewhere.}\end{cases}$$

If x_i; $i=1, 2, \cdots, n$; give the positions along the road of some ordered arrangement of the n persons, then the probability that $x_{i+1} \geqq x_i + b$; $i=1, 2, \cdots, n-1$; for this one arrangement of the n persons is given by the iterated integral,

$$I = \int_0^{a-(n-1)b} p(x_1)dx_1 \int_{x_1+b}^{a-(n-2)b} p(x_2)dx_2 \cdots \int_{x_{n-2}+b}^{a-b} p(x_{n-1})dx_{n-1} \int_{x_{n-1}+b}^{a} p(x_n)dx_n.$$

The evaluation of I is easy, giving

$$I = \left[\frac{a-(n-1)b}{a}\right]^n \Big/ n!.$$

Since there are $n!$ possible ordered arrangements of the n persons along the road, the desired probability is

$$P = n!I = [1-(n-1)b/a]^n.$$

Of course it is clear that the condition $(n-1)b \leqq a$ must be imposed.

Solved also by D. W. Alling, N. J. Fine, J. B. Kelly, G. W. Petrie, and the proposer.

Editorial Note. A slight variation of the problem is as follows:

There are n equal cylinders each of length c strung on a horizontal rod with stops at its ends which give a length of a on which the n cylinders can take various positions. For random positions of the cylinders what is the probability P_n that no two cylinders are less than b apart?

Denote by x_i the abscissa of the left end of the ith cylinder counting from the left stop, then we have the following inequalities

$$(i-1)(c+b) \leqq x_i \leqq x_{i+1} - (c+b), \qquad 1 \leqq i \leqq n-1;$$
$$(n-1)(c+b) \leqq x_n \leqq a - c.$$

If x_1 alone varies, then the measure of the favorable cases is $x_2-(c+b)$; if x_1, x_2 alone vary, the measure is $[x_2-2(c+b)]^2/2$; if x_1, x_2, x_3 alone vary, the measure is $[x_4-3(c+b)]^3/3!$; and so on up to and including the case where $x_1, x_2, \cdots, x_{n-1}$ alone vary and the measure is $[x_n-(n-1)(c+b)]^{n-1}/(n-1)!$. Finally if all vary, the measure is $[a-nc-(n-1)b]^n/n!$.

If no condition is imposed, then $b=0$, and the measure is now $[a-nc]^n/n!$; and we have

$$P_n = \left[1 - \frac{(n-1)b}{a-nc}\right]^n, \qquad a \geqq nc + (n-1)b.$$

Refinement of the Triangle Inequality

4171 [1945, 462]. *Proposed by Tibor Radó, Ohio State University*

Let $\mathbf{x}_n$, $n=0, 1, 2, \cdots$ be a sequence of vectors in euclidean three-space such that $|\mathbf{x}_n| > \delta$ for all n, where δ is a fixed positive constant, and absolute

value signs designate the length of the vector involved. Prove that the relation

$$|\mathbf{x}_n| + |\mathbf{x}_0| - |\mathbf{x}_n + \mathbf{x}_0| \to 0 \quad \text{as} \quad n \to \infty$$

holds if and only if there exists a sequence of positive scalars a_n such that $|a_n\mathbf{x}_n - \mathbf{x}_0| \to 0$ as $n \to \infty$.

Solution by R. C. Buck, Harvard University. Let $\mathbf{x}_n$ be a sequence of points of a normed linear space, for example, 3-space. We suppose that $\|\mathbf{x}_n\| \geqq \delta$ for all n.

First, if there is a sequence of positive scalars c_n such that $\lim \|c_n\mathbf{x}_n - \mathbf{x}_0\| = 0$, then $\lim \|\mathbf{x}_0\| + \|\mathbf{x}_n\| - \|\mathbf{x}_0 + \mathbf{x}_n\| = 0$. For, $\|\mathbf{x}_0\| = \|\mathbf{x}_0 - c_n\mathbf{x}_n + c_n\mathbf{x}_n\| \leqq c_n\|\mathbf{x}_n\| + \|\mathbf{x}_0 - c_n\mathbf{x}_n\|$ and $\|\mathbf{x}_0 + \mathbf{x}_n\| = \|(c_n+1)\mathbf{x}_n + \mathbf{x}_0 - c_n\mathbf{x}_n\| \geqq (c_n+1)\|\mathbf{x}_n\| - \|\mathbf{x}_n - c_n\mathbf{x}_n\|$ so that

$$0 \leqq \|\mathbf{x}_0\| + \|\mathbf{x}_n\| - \|\mathbf{x}_0 + \mathbf{x}_n\| \leqq 2\|\mathbf{x}_0 - c_n\mathbf{x}_n\| \to 0.$$

Now if, in addition, the space has an inner product, true of Euclidean n-space for example, then

$$\left\| \frac{\mathbf{x}}{\|\mathbf{x}\|} - \frac{\mathbf{y}}{\|\mathbf{y}\|} \right\|^2 = \frac{\|\mathbf{x}\| + \|\mathbf{y}\| + \|\mathbf{x} + \mathbf{y}\|}{\|\mathbf{x}\|\,\|\mathbf{y}\|} (\|\mathbf{x}\| + \|\mathbf{y}\| - \|\mathbf{x} + \mathbf{y}\|)$$

so that, with $\|\mathbf{x}_n\| \geqq \delta > 0$, and $\lim \|\mathbf{x}_0\| + \|\mathbf{x}_n\| - \|\mathbf{x}_0 + \mathbf{x}_n\| = 0$, we have

$$\lim \left\| \frac{\mathbf{x}_0}{\|\mathbf{x}_0\|} - \frac{\mathbf{x}_n}{\|\mathbf{x}_n\|} \right\| = 0$$

or, putting $\|\mathbf{x}_0\|/\|\mathbf{x}_n\| = c_n > 0$, $\lim \|\mathbf{x}_0 - c_n\mathbf{x}_n\| = 0$.

This remains true if the space lacks an inner product, but does satisfy a certain convexity relation, sufficient to infer the convergence of $\|\mathbf{x}_0/\|\mathbf{x}_0\| - \mathbf{x}_n/\|\mathbf{x}_n\|\|$ from that of $\|\mathbf{x}_0\| + \|\mathbf{x}_n\| - \|\mathbf{x}_0 + \mathbf{x}_n\|$. For example, we might require that for any positive δ there is an M such that for all $\mathbf{x}$, $\mathbf{y}$, $\|\mathbf{x}\| > \delta$, $\|\mathbf{y}\| > \delta$, we have

$$\left\| \frac{\mathbf{x}}{\|\mathbf{x}\|} - \frac{\mathbf{y}}{\|\mathbf{y}\|} \right\| \leqq M\{\|\mathbf{x}\| + \|\mathbf{y}\| - \|\mathbf{x} + \mathbf{y}\|\}.$$

Solved also by Truman Botts, Peter Frank, J. W. Green, P. C. Hammer, and Hans Samelson.

Finite Cosine Series

4172 [1945, 462]. *Proposed by R. P. Agnew, Cornell University*

Prove or disprove the following statement involving cosine series. If $a_1, a_2, a_3, \cdots$ is a sequence of real constants, and if L_n is the length of the part of the graph of the function

$$f_n(x) = a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx$$

lying in the interval $-\pi \leqq x \leqq \pi$ then $L_1 \leqq L_2 \leqq L_3 \leqq \cdots$.

I. *Solution by Fritz Herzog, Michigan State College.* The statement is not true. More specifically, for each $n > 2$ there exist values of $a_1, a_2, \cdots, a_n$ such

that $L_{n-1} > L_n$; however, $L_1 \leqq L_2$ for all values of a_1 and a_2.

Let n be a fixed integer >2. We consider L_n as a function of the n real variables $a_1, a_2, \cdots, a_n$. The statement that $L_{n-1} \leqq L_n$ would imply that if $a_1, a_2, \cdots, a_{n-1}$ are arbitrarily fixed L_n would have a minimum at $a_n = 0$. In order to prove that this cannot be true, it suffices to show that the expansion of L_n in a Taylor series about the origin ($a_1 = a_2 = \cdots = a_n = 0$) contains non-zero terms which are linear in a_n. We assume throughout the following that $a_1, a_2, \cdots, a_n$ are restricted to a sufficiently small neighborhood of the origin, for instance, the region defined by $|a_1| + 2|a_2| + \cdots + n|a_n| \leqq 1/2$.

We have

$$L_n = \int_{-\pi}^{\pi} [1 + (a_1 \sin x + 2a_2 \sin 2x + \cdots + na_n \sin nx)^2]^{1/2} dx,$$

$$L_n = \int_{-\pi}^{\pi} \sum_{r=0}^{\infty} \binom{1/2}{r} (a_1 \sin x + 2a_2 \sin 2x + \cdots + na_n \sin nx)^{2r} dx. \tag{1}$$

Expanding each term in the infinite series in (1) by the polynomial theorem, we obtain the expansion of the integrand in a Taylor series about the origin. It is easily seen that this Taylor series converges absolutely and uniformly for $-\pi \leqq x \leqq \pi$. Integrating term by term, we thus obtain from (1) the expansion of L_n itself in a Taylor series about the origin.

In order to show the presence in this Taylor series of non-zero terms, linear in a_n, let i, j, k be any integers with

$$0 < i \leqq j \leqq k < n. \tag{2}$$

The coefficient of $a_i a_j a_k a_n$ in the Taylor series for L_n is easily obtained from (1) as being equal to

$$\int_{-\pi}^{\pi} (-1/8) W (i \sin ix)(j \sin jx)(k \sin kx)(n \sin nx) dx$$

$$= -(Wijkn/32) \int_{-\pi}^{\pi} [\cos (j-i)x - \cos (j+i)x][\cos (n-k)x - \cos (n+k)x] dx,$$

where W equals 24 or 12 or 4, respectively, according to whether no two or exactly two or all three of the numbers i, j, k are equal to each other. By the use of the well-known orthogonality relations between the functions $\cos px$ ($p = 0, 1, 2, \cdots$) and since by (2) $n+k > j+i > j-i \geqq 0$, we conclude that the coefficient of the term $a_i a_j a_k a_n$, which by (2) is linear in a_n, will be different from zero if and only if

$$n - k = j - i \quad \text{or} \quad n - k = j + i. \tag{3}$$

Clearly, values of i, j, k satisfying (2) and (3) exist for each $n > 2$ (but not for $n = 2$). This concludes the proof of the first part of our statement.

To show that $L_1 \leqq L_2$ for all real values of a_1 and a_2, we make use of the fact that $(1+t^2)^{1/2}$ is a convex function of t and hence for real u and v

$$(4) \qquad [(1+u^2)^{1/2}+(1+v^2)^{1/2}]/2 \geqq \{1+[(u+v)/2]^2\}^{1/2}.$$

Put $u=a_1\sin x+2a_2\sin 2x$ and $v=a_1\sin x-2a_2\sin 2x$, so that $(u+v)/2=a_1\sin x$. Then integrating from $x=0$ to $x=\pi/2$, the left side of (4) yields $L_2/4$, while the right side gives $L_1/4$. Thus the inequality (4) leads to $L_2 \geqq L_1$.

Remark. An investigation of the signs of the coefficients of the terms $a_ia_ja_ka_n$ (see above) reveals the fact that for $n>2$ the statement $L_{n-1} \leqq L_n$ can be disproved even if the a_m are restricted to *non-negative* numbers.

II. *Solution by H. E. Bray, Rice Institute.* It will be proved that $L_1 \leqq L_2 \leqq L_3 \cdots$ is false. Let us define $f_1(x), f_2(x), f_3(x)$ as follows:

$$f_1(x) \equiv f_2(x) \equiv \cos x, \qquad f_3(x) \equiv \cos x + a\cos 3x,$$

where a is a parameter. Then L_3 is a function of a

$$L_3(a) = 2\int_0^{\pi} [1+(\sin x + 3a\sin 3x)^2]^{1/2}dx.$$

Note that $L_3(0)=L_2=L_1$. We shall prove that dL_3/da is positive when $a=0$. It will then follow that $L_3(a)$ is less than $L_3(0)=L_2=L_1$ if a is negative and numerically small. We have, after differentiating under the integral sign and setting $a=0$,

$$\begin{aligned} dL_3/da\,|_{a=0} &= 6\int_0^{\pi} \sin x \sin 3x(1+\sin^2 x)^{-1/2}dx \\ &= 6\int_0^{\pi/2} (\cos 2x - \cos 4x)(1+\sin^2 x)^{-1/2}dx \\ &= 6\int_0^{\pi/2} g(x)(1+\sin^2 x)^{-1/2}dx. \end{aligned}$$

In the last integral $g(x) \equiv \cos 2x - \cos 4x$. The function $g(x)$ is positive, $0<x<\pi/3$, and is negative, $\pi/3<x<\pi/2$. Moreover $\int_0^{\pi/2} g(x)dx=0$. But $(1+\sin^2 x)^{-1/2}$ is a positive decreasing function, $0 \leqq x \leqq \pi/2$, which takes on the value $2/\sqrt{7}$ when $x=\pi/3$. Therefore

$$\int_0^{\pi/3} g(x)(1+\sin^2 x)^{-1/2}dx > \frac{2}{\sqrt{7}}\int_0^{\pi/3} g(x)dx$$

$$\int_{\pi/3}^{\pi/2} g(x)(1+\sin^2 x)^{-1/2}dx > \frac{2}{\sqrt{7}}\int_{\pi/3}^{\pi/2} g(x)dx.$$

Hence, on adding, we obtain the inequality

$$(1/6)dL_3/da]_{a=0} = \int_0^{\pi/2} g(x)(1+\sin^2 x)^{-1/2}dx > \frac{2}{\sqrt{7}}\int_0^{\pi/2} g(x)dx = 0$$

from which it follows that dL_3/da is positive when $a=0$. Thus $L_3(a)<L_2=L_1$ when a is negative and numerically small.

Solved also by the proposer.

The proposer stated that it can be shown that $L_1 \leqq L_2$, with equality only when $a_2=0$; and to show that the statement is not true he considered $a_1=1$, $a_2=0$, $a_3=c/3$, and proceeded as in Bray's elegant solution to show that $[dL_3/dc]_{c=0}>0$.

Twelve Point Sphere

4175 [1945, 463]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The twelve point sphere of any tetrahedron is the locus of the points such that the sum of the squares of their distances to the vertices diminished by their powers with respect to the circumsphere, is equal to a third of the sum of the squares of the edges.

Solution by the Proposer. The following generalization will be proved.

Theorem. The locus of the points such that the sum of the squares of their distances to the vertices of a tetrahedron diminished by their powers with respect to its cicumsphere is constant, is a sphere concentric to the twelve point sphere of the tetrahedron.

Proof. Given the tetrahedron $ABCD$, set $BC=a$, $DA=a'$, $CA=b$, $DB=b'$, $AB=c$, $DC=c'$; denote by O and R the circumcenter and circumradius, and by G the centroid. Then the variable point M is such that

$$(1)\qquad (MA)^2+(MB)^2+(MC)^2+(MD)^2-[(MO)^2-R^2]=k\Sigma[a^2+(a')^2],$$

where the right member is the given constant. Since

$$\begin{aligned}(MA)^2+(MB)^2+(MC)^2+(MD)^2&=(GA)^2+(GB)^2+(GC)^2+(GD)^2+4(GM)^2\\&=\tfrac{1}{4}\Sigma[a^2+(a')^2]+4(GM)^2,\end{aligned}$$

the relation (1) can be written

$$4(GM)^2-(OM)^2=-R^2+(k-\tfrac{1}{4})\Sigma[a^2+(a')^2],$$

and this represents a sphere (S) with the center ω such that $\omega G/\omega O=1/4$, and ω coincides with the center of the twelve point sphere of $ABCD$. The square of the radius is equal to

$$(\omega M)^2=\frac{R^2}{9}+\tfrac{1}{9}(k-\tfrac{1}{4})\Sigma[a^2+(a')^2],$$

and conversely. The Stewart theorem applied to triangle $MO\omega$ gives in fact

$$4(GM)^2-(OM)^2=3(\omega M)^2-\tfrac{1}{3}(OG)^2,$$

$$(OG)^2=R^2-\tfrac{1}{16}\Sigma[a^2+(a')^2].$$

For the particular case of the problem where $k=1/3$, the sphere (S) is the twelve point sphere with the radius $R/3$.

Solved also by W. T. Short.

Editorial Note. The solution by Short referred to Court's *Modern Pure Geometry*, p. 251, for theorems as to the twelve point sphere, and used a system of coordinates in the proof.

The generalized theorem can be extended to a simplex in euclidean space of n dimensions as follows. The $3(n+1)$ point sphere of a simplex S in euclidean space of n dimensions is the sphere (L) through the $n+1$ centroids of its faces; if M is the Monge point of S it passes through the $n+1$ points A_i' where $MA_i' = MA_i/n$ where A_i is a vertex of S; and it also passes through the points A_i'' the projections of A_i' on the corresponding faces opposite to A_i. The center of the $3(n+1)$ point sphere is at the point L where $ML = MC/n$, C being the center of the circumsphere (C) of S; and its radius is R/n, R being the radius of (C). The sphere (L) is the inverse of (C) with respect to the polar sphere of S.

Thus if M is taken as the origin of vectors, $\mathbf{a}_i$, $\mathbf{c}$, $\mathbf{g}$ being the vectors of A_i, C, G, the last being the centroid of S, the vector equation of (L) is

$$(L); \qquad \left(\mathbf{x} - \frac{\mathbf{c}}{n}\right)^2 = \frac{R^2}{n^2}, \qquad \mathbf{x}^2 - \frac{2\mathbf{c}\cdot\mathbf{x}}{n} + \frac{\mathbf{c}^2 - R^2}{n^2} = 0.$$

The polar sphere of S has the equation $\mathbf{x}^2 - \rho = 0$, and the equation of (L) may also be written

$$(L) \equiv (C'); \qquad n\mathbf{x}^2 - 2\mathbf{c}\cdot\mathbf{x} + \rho = 0, \quad n\rho = \mathbf{c}^2 - R^2, \quad 2\mathbf{c} = (n+1)\mathbf{g}.$$

Let P be a point such that

$$\Sigma\,(PA_i)^2 - [(CP)^2 - R^2] = k \text{ (constant)}.$$

Then the equation of the locus of P is

$$(n+1)\mathbf{x}^2 - 4\mathbf{c}\cdot\mathbf{x} + \Sigma\,\mathbf{a}_i^2 - [\mathbf{x}^2 - 2\mathbf{c}\cdot\mathbf{x} + \mathbf{c}^2 - R^2] = k,$$
$$n\mathbf{x}^2 - 2\mathbf{c}\cdot\mathbf{x} + [\Sigma\,\mathbf{a}_i^2 - \mathbf{c}^2 + R^2 - k] = 0.$$

Hence the locus of P is a sphere with center at L given by the vector $\mathbf{c}/n$. To find its radius we write the locus equation

$$\left(\mathbf{x} - \frac{\mathbf{c}}{n}\right)^2 = \{(n+1)\mathbf{c}^2 + n[k - R^2 - \Sigma\,\mathbf{a}_i]\}/n^2$$
$$= \frac{1}{n^2}[(n-1)^2\mathbf{c}^2 - n(n+2)R^2 + nk],$$

and it follows that the square of its radius is the right member. For a suitable value of k the locus reduces to (L), and to determine its value we solve

$$(n-1)^2\mathbf{c}^2 - n(n+2)R^2 + nk = R^2$$
$$nk = (n+1)^2R^2 - (n-1)^2\mathbf{c}^2 = \Sigma\,e_{ij}^2 = \Sigma\,(A_iA_j)^2.$$

In the Editorial Note for 4049 [1943, 577] there are given a number of vector equations related to the simplex S where m is used in place of the above ρ.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

The first three graduate scholarships granted by the Sperry Gyroscope Company have been awarded to E. J. Barlow, O. T. Schultz, and W. E. Tinsley.

Michigan State College announces the following appointments: Dr. J. H. Bell of the University of Manitoba, Professor H. L. Black of Westminster College, Dr. G. M. Brown of the Harrow School, England, and Dr. Leo Katz of Wayne University have been appointed to assistant professorships; Assistant Professor C. P. Wells has been promoted to an associate professorship.

Oklahoma Agricultural and Mechanical College makes the following announcements: Dr. J. C. C. McKinsey has been appointed to an associate professorship; Assistant Professors P. E. Lewis and C. E. Marshall have been promoted to associate professorships; R. L. Caskey, N. D. Griffin, and W. L. Shepherd have been promoted to assistant professorships.

Stanford University announces: Associate Professors George Polya, A. C. Schaeffer and D. C. Spencer have been promoted to professorships.

The United States Naval Academy makes the following announcements: Dr. J. C. Abbott and Dr. C. W. Seekins have been appointed to assistant professorships, Dr. E. E. Betz, Dr. T. J. Benac and R. C. Morrow have been promoted to assistant professorships, Dr. C. P. Brady and Dr. T. L. Downs have been appointed to associate professorships. Assistant Professor T. W. Moore has been promoted to an associate professorship. At the Postgraduate School, Assistant Professor Randolph Church has been promoted to an associate professorship and Associate Professor C. C. Torrance of the Case School of Applied Science has been appointed to an associate professorship.

The University of Michigan makes the following announcements: Associate Professor P. S. Dwyer has been promoted to a professorship, Assistant Professors C. J. Nesbitt and E. D. Rainville have been promoted to associate professorships, Assistant Professor Nathaniel Coburn of the University of Texas, Dr. Isaac Opatowski of the University of Chicago, Assistant Professor M. O. Reade of Purdue University and Assistant Professor Hans Samelson of Syracuse University have been appointed to assistant professorships, and Dr. C. L. Dolph has been appointed to a lectureship. Assistant Professor W. A. Ambrose has been granted a leave of absence to serve as Visiting Assistant Professor at Yale University. Professor Peter Field has retired.

The University of Wyoming makes the following announcements: Professor O. H. Rechard has resigned as head of the department to become dean of the

liberal arts college, Professor C. F. Barr has been appointed acting head of the department, Dr. Leonard Bristow, Dr. S. R. Smith and Palmer Steen have been appointed to assistant professorships, Assistant Professor Verne Varineau has returned to the university and Dr. Nathan Schwid has been appointed to an associate professorship.

Associate Professor H. E. Arnold of Wesleyan University has been promoted to a professorship.

Dr. Winifred A. Asprey of Vassar College has been promoted to an assistant professorship.

Dr. S. P. Avann of Oregon State College has been appointed to an assistant professorship at the University of Washington.

Associated Professor H. W. Bailey of the University of Illinois has been appointed Associate Dean for Liberal Arts and Sciences in the Division of Undergraduate Studies of the University of Illinois, Chicago, Illinois.

Assistant Professor J. J. Barron of Marquette University has been promoted to an associate professorship.

Assistant Professor A. E. Basch of Amherst College has been appointed to an associate professorship at the Massachusetts State College at Fort Devens.

J. K. Baumgart has been appointed to an assistant professorship at Elmhurst College, Elmhurst, Illinois.

Dr. Dorothy L. Bernstein of the University of Rochester has been promoted to an assistant professorship.

Dr. E. E. Blanche has accepted the position of principal administrative analyst, Management Branch, Plans and Policy Office, Service, Supply and Procurement Division of the War Department.

Associate Professor M. G. Boyce at Western Reserve University has been appointed to an associate professorship at Vanderbilt University.

Dr. G. W. Brown has been appointed to an associate professorship at the Iowa State College of Agriculture and Mechanical Arts.

Professor C. T. Bumer of Kenyon College has been appointed Head Mathematician, Assistant to the Director of the Evaluation and Analysis Group, Bureau of Ordnance.

Assistant Professor H. E. Burns of Calumet Extension Center, Indiana University, has been appointed director.

K. A. Bush of the United States Naval Academy has been appointed to an assistant professorship at Mohawk College, Utica, New York.

Assistant Professor F. A. Butter, Jr., of Stanford University has been appointed mathematical consultant at the Hughes Aircraft Company, Culver City, California.

Assistant Professor H. H. Campaigne of the University of Minnesota has accepted a position as mathematician with the Navy Department.

Dr. G. F. Carrier has been appointed to an assistant professorship at Brown University.

Dr. A. B. Carson has been appointed to an associate professorship at the Army Air Force Institute of Technology, Wright Field, Dayton, Ohio.

Associate Professor J. W. Cell of the State College of Agriculture and Engineering of the University of North Carolina has been promoted to a professorship.

Assistant Professor C. L. Clark of Oregon State College has been promoted to an associate professorship.

Dr. W. W. S. Claytor has been appointed to a professorship at Hampton Institute.

Assistant Professor A. H. Clifford of the Massachusetts Institute of Technology has been appointed to an associate professorship at Johns Hopkins University.

Professor Lennie P. Copeland of Wellesley College has retired with the title professor emeritus.

Dr. Byron Cosby, Jr., of the United States Naval Academy has been appointed to an assistant professorship at the University of Iowa.

Professor C. C. Craig of the University of Michigan has been appointed director of the Statistical Research Laboratory.

Dr. W. J. R. Crosby of the University of Saskatchewan has been promoted to an assistant professorship.

Dr. E. L. Crow has accepted a position as mathematician at the United States Naval Ordnance Test Station, Inyokern, California.

Dr. J. F. Daly has accepted a position as statistician with the Bureau of the Census.

Dr. Margaret R. Davis has been appointed to an associate professorship at Southeastern Louisiana College, Hammond, Louisiana.

Professor Charlotte I. Davison of Wilson College, Chambersburg, Pennsylvania, has retired with the title professor emeritus. She received the honorary degree of Doctor of Humane Letters from Wilson College in June, 1946.

Dr. W. W. Denton has been appointed principal of the High School, Minden City, Michigan.

Associate Professor Douglas Derry of the University of Saskatchewan has been appointed to an associate professorship at the University of British Columbia.

Dr. J. B. Diaz of Brown University has been appointed to an assistant professorship at Carnegie Institute of Technology.

Sister Agnes T. Dimond of the College of Mount Saint Joseph, Mount Saint Joseph, Ohio, has been promoted to a professorship.

Professor Emeritus L. L. Dines of Carnegie Institute of Technology has been appointed to a visiting professorship at Smith College.

Professor J. E. Dotterer of Manchester College, North Manchester, Indiana, has been appointed professor of mathematics and head of the department.

Dr. D. M. Dribin of the Army Security Agency has been awarded the Legion of Merit.

J. A. H. Duffie has been appointed to a professorship in physics at Assumption College, University of Western Ontario.

Dr. J. N. Eastham of Nazareth College has been appointed to an assistant professorship at the Cooper Union School of Engineering.

Assistant Professor F. G. Fender of Rutgers University has been promoted to a professorship.

Associate Professor E. J. Finan of Catholic University has been promoted to a professorship.

Dr. N. J. Fine has accepted a position with the Operations Evaluation Group, Navy Department.

Assistant Professor F. G. Fisher of the University of Santa Clara has been promoted to an associate professorship.

J. D. Fitzpatrick has been appointed to an assistant professorship at Marquette University.

N. S. Free of Mount Royal College, Calgary, Alberta, Canada, has been appointed to a lectureship at the University of British Columbia.

Dr. Bernard Friedman has been appointed to an assistant professorship at New York University.

Dr. G. N. Garrison of the College of the City of New York has been appointed to an assistant professorship at Lehigh University.

Dr. R. E. Gaskell of Brown University has been appointed to an assistant professorship at the Iowa State College of Agricultural and Mechanical Arts.

Associate Professor F. C. Gentry of the Louisiana Polytechnic Institute has been promoted to a professorship.

M. E. Gillis of the University of Tennessee has been appointed to an acting associate professorship at the University of Florida.

W. H. Glenn of Pasadena Junior College has been appointed head of the division of natural science of Muir Junior College, Pasadena, California.

Professor V. D. Gokhale of the University of the Philippines has been appointed to a visiting lectureship at Rutgers University.

Dr. H. H. Goldstine of the University of Michigan has been promoted to an assistant professorship.

Assistant Professor Michael Golomb of Purdue University has been promoted to an associate professorship.

Assistant Professor S. H. Gould of Victoria College, University of Toronto, has been promoted to an associate professorship.

Dr. A. A. Grau has been appointed to an assistant professorship at the University of Kentucky.

Dr. R. E. Greenwood of the University of Texas has been promoted to an assistant professorship.

Professor W. C. Griffith of Marion Institute, Marion, Alabama, has been appointed to an associate professorship at Centenary College, Shreveport, Louisiana.

Dr. N. A. Hall has been appointed head of the Analysis Section, Research Department, United Aircraft Corporation.

Assistant Professor P. C. Hammer of Oregon State College has been promoted to an associate professorship.

Dr. D. C. Harkin of Brooklyn College has been appointed mathematical consultant at the Naval Research Laboratory.

Assistant Professor F. S. Harper of the University of Nebraska has been appointed to a professorship at Drake University, Des Moines, Iowa.

Assistant Professor O. G. Harrold of Pomona College has been appointed lecturer at Princeton University.

Dr. Philip Hartman has been appointed to an assistant professorship at Johns Hopkins University.

Cecil Hastings has been appointed research analysist with the Douglas Aircraft Company, Santa Monica, California.

Dr. Charles Hatfield, Jr., has been appointed to an assistant professorship at the University of Minnesota.

Professor E. K. Haviland of Lincoln University, Lincoln University, Pennsylvania, has been appointed to an assistant professorship at Johns Hopkins University.

Assistant Professor Euphemia L. Haynes of Miner Teachers College, Washington, D. C., has been promoted to an associate professorship.

Manuel Herschdorfer of Amherst College has been appointed to an assistant professorship at Seton Hall College, South Orange, New Jersey.

J. S. Hickman of Rochester Junior College has been appointed physicist at the United States Naval Electronics Laboratory, San Diego, California.

Assistant Professor E. H. C. Hildebrandt of Northwestern University has been promoted to an associate professorship.

Dr. V. A. Hoersch of the University of Illinois has been promoted to an assistant professorship.

Associate Professor Aughtum S. Howard of Kentucky Wesleyan College, Winchester, Kentucky, has been promoted to a professorship.

Dr. L. C. Hutchinson of the Norden Laboratories has been appointed to an assistant professorship at the Polytechnic Institute of Brooklyn.

Professor J. B. Jackson of the University of South Carolina has been appointed Dean of Men.

A. W. Jacobson has been promoted to an assistant professorship at Wayne University.

Assistant Professor W. C. Janes of the Kansas State College of Agriculture and Applied Science has been promoted to an associate professorship.

J. B. Jeffries has been appointed to a professorship at the Agricultural and Technical College of North Carolina.

Walter Jennings has been appointed to an assistant professorship at Virginia Polytechnic Institute.

Assistant Professor Dora E. Kearney of Iowa State Teachers College has resigned.

J. R. F. Kent has been appointed to an assistant professorship at the University of British Columbia.

Dr. F. L. Kiokemeister of Purdue University has been appointed to an assistant professorship at Mount Holyoke College.

Dr. E. A. Knobelauch of the University of Pennsylvania has been appointed lecturer at Rutgers University.

Dr. G. B. Lang of the University of Florida has been promoted to an assistant professorship.

Associate Professor H. D. Larsen of the University of New Mexico has been promoted to a professorship.

H. R. Leifer has accepted a position as assistant chief, Registration and Research Section, Veterans Administration, Pittsburgh, Pennsylvania.

Associate Professor Hans Lewy of the University of California has been promoted to a professorship.

Dr. F. W. Light of the Johns Hopkins University has been promoted to an assistant professorship.

Assistant Professor H. W. Linscheid of Eastern New Mexico College has been appointed to an associate professorship at the College of Emporia, Emporia, Kansas.

Associate Professor Mayme I. Logsdon of the University of Chicago has retired. During the academic year 1946–47 she will be visiting professor at the University of Miami.

Dr. L. H. Loomis of Harvard University has been promoted to an associate professorship.

R. A. Lytle of the University of Virginia has been appointed to an adjunct professorship at the University of South Carolina.

Associate Professor J. D. Mancill of the University of Alabama has been promoted to a professorship.

Ralph Mansfield of Weidenhoff, Inc., Chicago, Illinois, has been promoted to the rank of chief engineer.

Associate Professor Morris Marden of the University of Wisconsin at Milwaukee has been promoted to a professorship.

Dr. A. J. Maria of Brooklyn College has been promoted to an assistant professorship.

Dr. A. E. May of the University of Wisconsin at Kenosha has been promoted to an assistant professorship.

W. S. McCulley of the Agricultural and Mechanical College of Texas has been promoted to an assistant professorship.

Dr. Janet McDonald of Vassar College has been promoted to an assistant professorship.

Professor W. H. McEwen of Mt. Allison University has been appointed to a professorship at the University of Manitoba.

Dr. A. W. McGaughey of the United States Naval Academy has been appointed to a professorship at Westminster College, New Wilmington, Pennsylvania.

P. E. Meadows has been appointed to an assistant professorship at Washington and Lee University.

Professor H. A. Meyer of Indiana University has been appointed to an associate professorship at the University of Florida.

Associate Professor Herman Meyer of the University of Miami has been promoted to a professorship.

Assistant Professor W. I. Miller of Bucknell University has been promoted to an associate professorship.

Dr. H. J. Miser has been appointed to an assistant professorship at Williams College.

Associate Professor Josephine M. Mitchell of Winthrop College has been appointed to an associate professorship at the Texas State College for Women.

E. C. Molina, formerly with Bell Telephone Laboratories, has been appointed special lecturer at Newark College of Engineering.

Dr. L. J. Monville of John Carroll University has been promoted to an assistant professorship.

Dr. C. W. Moran of Illinois Institute of Technology has accepted a position at Wright Junior College.

Professor Eugenie M. Morenus of Sweet Briar College has retired.

Sister M. Joanne Muggli of Saint Benedict's College, Saint Joseph, Minnesota, has been promoted to an assistant professorship.

Assistant Professor David Nelson of Amherst College has been appointed to an assistant professorship at George Washington University.

Dr. Albert Newhouse of the University of Houston has been promoted to an assistant professorship.

Professor R. E. Norris of State Teachers College, Milwaukee, Wisconsin, has been appointed dean of instruction.

Associate Professor E. P. Northrop of the University of Chicago has been appointed associate dean of the College.

Assistant Professor Emma J. Olson of Kent State University, Kent, Ohio, has been promoted to an associate professorship.

Dr. H. L. Olson of Western Union College, Le Mars, Iowa, has been appointed to a professorship at Indiana Technical College, Fort Wayne, Indiana.

R. I. Pepper has been appointed to an assistant professorship at Winthrop College, Rock Hill, South Carolina.

Dr. A. S. Peters of New York University has been promoted to an assistant professorship.

Associate Professor H. R. Phalen of the College of William and Mary has been promoted to a professorship.

Associate Professor R. J. Pitts of Fort Valley State College, Fort Valley, Georgia, has been promoted to a professorship.

Associate Professor H. S. Pollard of Miami University has been promoted to a professorship.

Assistant Professor A. R. Poole of Montana State College has been appointed to an assistant professorship at Oregon State College.

Assistant Professor E. S. Quade of the University of Florida has been promoted to an associate professorship.

L. J. Quaid of the University of Minnesota has been promoted to an assistant professorship.

Ellen F. Rasor has been appointed professor of mathematics and head of the department at Flora Macdonald College, Red Springs, North Carolina.

Assistant Professor H. W. Raudenbush of Queens College, Flushing, New York, has been promoted to an associate professorship.

Reverend John Raymond of Saint Martin's College, Lacey, Washington, has been promoted to a professorship.

Assistant Professor L. M. Reagan of the Polytechnic Institute of Brooklyn has been appointed to an assistant professorship at the University of Wichita.

Professor L. J. Reed of Johns Hopkins University has been appointed vice president.

Professor P. K. Rees of Southwestern Louisiana Institute, Lafayette, Louisiana, has been appointed to an associate professorship at Louisiana State University.

Assistant Professor Eric Reissner of the Massachusetts Institute of Technology has been promoted to an associate professorship.

Dr. C. E. Rickart of Yale University has been promoted to an assistant professorship.

Assistant Professor R. M. Robinson of the University of California has been promoted to an associate professorship. He will be a visiting lecturer at Princeton University this academic year.

Associate Professor A. E. Ross of Saint Louis University has been appointed to a professorship at the University of Notre Dame.

S. G. Roth of Cooper Union has been appointed to an assistant professorship at New York University.

Assistant Professor Raphael Salem of the Massachusetts Institute of Technology has been promoted to an associate professorship.

Associate Professor George Sauté of Rollins College has been promoted to a professorship.

Dr. Alice T. Schafer has been appointed lecturer at New Jersey College for Women, Rutgers University.

Dr. W. T. Scott of Northwestern University has been promoted to an assistant professorship.

Assistant Professor C. E. Sealander of the University of South Dakota has been appointed to an assistant professorship at the Iowa State College of Agriculture and Mechanical Arts.

M. Anice Seybold of the University of Illinois has been appointed to an assistant professorship at North Central College, Naperville, Illinois.

Dr. M. E. Shanks of the University of Missouri has been appointed to an assistant professorship at Purdue University.

Associate Professor D. T. Sigley of Kansas State College has been appointed research engineer, Institute of Cooperative Research, Johns Hopkins University.

Professor Heinz Simon of Southern Union College has been appointed to a professorship at William Penn College, Oskaloosa, Iowa.

Harold E. Smith of the Junior College of Connecticut has been promoted to an assistant professorship.

Dr. Abraham Spitzbart of the University of Wisconsin has been appointed to an assistant professorship at the University of Wisconsin in Milwaukee.

Assistant Professor D. W. Starr of Southern Methodist University has been promoted to an associate professorship.

Dr. J. C. Stewart of Louisiana State University has been appointed to an assistant professorship at Lawrence College, Appleton, Wisconsin.

Assistant Professor R. R. Stoll of Williams College has been appointed to an assistant professorship at Lehigh University.

Irving Sussman of Johns Hopkins University has been appointed to an assistant professorship at the University of Dayton.

Dr. Ethel Sutherland of East Carolina Teachers College has been appointed to an associate professorship at the State Teachers College, Farmville, Virginia.

Associate Professor H. S. Thurston of the University of Alabama has been promoted to a professorship.

Dr. C. B. Tompkins has been appointed director of research, Engineering Research Associates, Incorporated, Saint Paul, Minnesota.

Dr. G. B. Van Schaack has been appointed to an assistant professorship at Union College, Schenectady, New York.

Dr. T. C. G. Wagner has been appointed to an associate professorship at the University of Maryland.

Associate Professor O. E. Walder of South Dakota State College has been promoted to a professorship.

Dr. P. R. Wallace has been appointed to an associate professorship at McGill University.

Assistant Professor R. M. Walter of New Jersey College for Women, Rutgers University, has been promoted to an associate professorship.

Dr. L. F. Walton of the University of California at Santa Barbara has been promoted to an assistant professorship.

Associate Professor L. E. Ward of the State University of Iowa has been appointed mathematician with the Aircraft Fire Control Section, Research, Development, and Test Organization, Naval Ordnance Test Station, Inyokern, California.

Associate Professor W. G. Warnock of the University of Alabama has been appointed to an associate professorship at Rensselaer Polytechnic Institute.

Dr. J. V. Wehausen has been appointed mathematician at the David Taylor Model Basin.

Professor A. H. Wheeler of Clark University has retired.

Dr. Wilfred Wilson of the University of Illinois has been promoted to an assistant professorship.

Dr. Y. K. Wong of the University of North Carolina has been promoted to an associate professorship.

Associate Professor C. B. Wright of East Texas State Teachers College, Commerce, Texas, has been promoted to a professorship.

Assistant Professor C. R. Wylie of Ohio State University has been appointed to a professorship at the Army Air Force Institute of Technology, Wright Field, Dayton, Ohio.

Assistant Professor S. D. Zeldin of the Massachusetts Institute of Technology has been promoted to an associate professorship.

Sister M. Claudia Zeller of the College of Saint Francis has been promoted to an assistant professorship.

The following appointments to instructorships are announced:

Agricultural and Mechanical College of Texas: W. E. Beeman, J. A. Daum, N. W. Wells

Brooklyn College: Dr. H. S. Kieval

College of the City of New York: J. H. Blau, Dr. Solomon Hurwitz, Herbert Solomon

Cooper Union: M. E. Levenson

Cornell University: Dr. Y. H. Kuo

Creighton University: Helen E. Clarkson

De Paul University: Rev. F. J. Fischer

Michigan State College: Evelyn O. Bychinsky, Laura J. Bullard, Frances E. Davis, H. A. Hanson, Emerson Grindall, H. H. Holloway, C. A. Jacokes, M. S. Miller; in physical science: J. A. Straw

Montana State College: Dr. H. M. Schaerf

Newark College of Arts and Sciences, Rutgers University: J. J. McCarthy

New York University: S. D. Bernardi

Oklahoma Agricultural and Mechanical College: Milton Berg, J. L. Cawood, Helen Dayton, R. D. Morrison, P. C. Stanger

Pennsylvania State College: Helen F. Story

Purdue University: Dr. Lee Byrne

Rutgers University: Dr. L. M. Court, Luis Nanni, Dr. M. A. Hyman, Aaron Galuten

Rice Institute: Dr. H. D. Brunk

Stanford University: A. V. Baez

United States Naval Academy: A. R. Craw, G. J. Mann, Dr. R. C. Rand, Dr. H. K. Sohl, W. J. Strange, E. G. Swafford

The University of California: Dr. C. A. Hayes, Dr. Alfred Horn, Dr. E. L. Lehmann
The University of Chicago: R. S. Fouch, Dr. Daniel Zelinsky
University of Cincinnati: H. D. Lipsich
The University of Michigan: Dr. W. M. Kincaid
University of Pennsylvania: Dr. I. S. Cohen
University of Rochester: Dr. Horace Komm
The University of Wyoming: Rosamond Jones
Washington University: Marlow Sholander, Dr. W. J. Thron
Wellesley College: Kathleen E. Butcher

A. M. Freeman, director of the mathematical laboratory, Boston Fiduciary and Research Association, died May 20, 1946.

Professor Emeritus Dean Townley of Stanford University died March 16, 1946.

The Editor-in-Chief wishes to express his indebtedness to the following persons who have served as referees of papers during 1945 and 1946.

C. B. Allendoerfer, I. A. Barnett, E. F. Beckenbach, Garrett Birkhoff, Henry Blumberg, L. M. Blumenthal, W. C. Brenke, R. H. Cameron, W. B. Caton, F. W. Cheney, R. V. Churchill, Nathaniel Coburn, N. B. Conkwright, J. L. Coolidge, A. H. Copeland, H. S. M. Coxeter, H. V. Craig, John DeCicco, Otto Dunkel, Paul Erdös, G. C. Evans, C. J. Everett, Howard Eves, Tomlinson Fort, J. S. Frame, Philip Franklin, Orrin Frink, Jr., Wallace Givens, J. W. Green, Lois W. Griffiths, P. R. Halmos, M. L. Hartung, G. E. Hay, C. G. Hempel, T. R. Hollcroft, Dunham Jackson, R. A. Johnson, Irving Kaplansky, L. M. Kells, R. E. Langer, D. H. Lehmer.

C. C. MacDuffee, W. T. Martin, C. W. Mendel, F. D. Murnaghan, J. R. Musselman, C. V. Newsom, Ivan Niven, E. P. Northrop, Rufus Oldenburger, Gordon Pall, B. C. Patterson, G. B. Price, Hans Rademacher, Tibor Radó, E. D. Rainville, Haim Reingold, Eric Reissner, M. M. Resnikoff, P. R. Rider, E. D. Schell, O. F. G. Schilling, I. M. Sheffer, T. L. Wade, R. J. Walker, H. S. Wall, Alexander Weinstein, Marie J. Weiss, A. L. Whiteman, D. V. Widder, L. R. Wilcox, John Williamson, and Roscoe Woods.

GENERAL INFORMATION

EDITED BY C. V. NEWSOM

Send information of especial interest to mathematicians, exclusive of personal items, to C. V. Newsom, Oberlin College, Oberlin, Ohio.

THE ARMY UNIVERSITIES IN EUROPE: A PREVIEW OF THE VETERANS PROBLEM.*

H. M. GEHMAN, University of Buffalo

1. *The Army Education Program in Europe.* After the end of the fighting in Europe in the spring of 1945, many of the soldiers who had fought there were eligible for discharge under the Army's point system and were to be returned to this country as soon as shipping was available. It was evident, however, that many men would have to wait for months in Europe before facilities were available for bringing them home. The Army Education Program was instituted to keep these men occupied, to help them make the transition from military life to civilian life, and in particular to give those who intended to enter educational institutions after discharge a chance to begin their studies while still in the Army.

The program was a comprehensive one. At the apex stood the University Centers, with which this paper is concerned. There were three of these, located at Florence, Italy; at Biarritz, France; and at Shrivenham, England.

The Universities at Shrivenham and Biarritz were planned as comprehensive educational institutions. Each contained eight sections (corresponding to schools in the ordinary civilian university): Agriculture, Commerce, Education, Engineering, Fine Arts (Music and Art), Journalism, Liberal Arts, and Science. Each section in turn was divided into branches (corresponding to departments). The Science section consisted of the five branches: Biology, Chemistry, Geology, Mathematics, and Physics.

2. *Recruiting the faculties.* In the original planning of the Army University Centers, it was contemplated that the faculty would consist largely of civilians brought to Europe just for this purpose. Although many qualified instructors were already in the armed forces, it was difficult to obtain the transfer of enough to staff the universities.

The recruiting of the civilian faculty was done from the Pentagon Building in Washington during June, 1945. I spent most of that month there endeavoring to secure instructors in mathematics for Shrivenham and Biarritz. The qualifications sought by the Army and the restrictions imposed were: Only men with experience in college teaching were desired, and it was hoped to secure men with reputations as outstandingly good teachers. Instructors were to be men, aged

* A portion of an address delivered to the Mathematical Association of America at Ithaca, New York, on August 19, 1946.

42 or over. A physical examination was required, but it was not a very rigorous one. Of course, good character was required. Finally it was necessary that each man be able to obtain a release from his institution and from his family without imposing too many hardships upon either. Since it was contemplated that the universities at Shrivenham and Biarritz would continue for a year, each instructor was asked to agree to stay with the Army for at least seven months, and those who so desired were given contracts for a year. The salary arrangements were liberal, so that no one would suffer financially by accepting a position overseas.

Of those invited to join the mathematics departments of the universities at Shrivenham and Biarritz, about one-third accepted appointments and went overseas during July. Of those who refused, a few could not accept for personal reasons. In most cases, the individual could not be spared by his institution because its mathematics department had already been depleted by leaves of absence for work in connection with the war effort.

The mathematics departments at Shrivenham and Biarritz were among the leading departments in their respective institutions with respect to teaching ability and scientific standing. With one or two exceptions, the civilian instructors in mathematics were full professors, holders of the Ph.D. degree, and members of the Society and the Association. Many were department heads; many were authors of textbooks and contributors to mathematical periodicals; two were members of the editorial staff of this MONTHLY. The military members of the mathematics departments were for the most part younger men and consequently had had less in the way of teaching experience and scientific training, but all those with whom I was acquainted were adequately prepared for the teaching which they were doing.

3. *Shrivenham American University.* The university center in England was located at Shrivenham Barracks, in the village of Shrivenham, Berkshire, about 75 miles due west of London. It was officially known in the beginning as Army University Center Number 1 (AUC #1). Later its name was changed to Shrivenham American University (SAU).

Housing facilities were adequate at Shrivenham, which was one reason for its selection as the site of the university center. The buildings on the post had been erected during the period from 1937 to 1939, with the exception of a 200-year old manor house which was used as the officers' club and mess. The faculty were housed in small brick residences.

There was a shortage of classrooms, and most of the classrooms were small. As for equipment, blackboards were literally boards painted black. Erasers were made by nailing strips of felt to blocks of wood. In the beginning there was a shortage of chalk.

Fortunately there was a sufficient supply of textbooks in mathematics. There were available the Education Manuals,* which are standard textbooks bound in paper covers, and also many textbooks that had been used in the ASTP

* These books are described in this MONTHLY, vol. 52, 1945, pp. 288–289.

courses. After the close of the ASTP, these books had been shipped to England in anticipation of their use in the Army Education Program.

Courses in Mathematics offered at Shrivenham included the usual undergraduate courses: Intermediate Algebra, College Algebra, Plane Trigonometry, Spherical Trigonometry, Plane Analytic Geometry, Differential Calculus, Integral Calculus, Statistics, Advanced Statistics, Astronomy, Review of Engineering Mathematics, Higher Mathematics for Engineers and Physicists, and Theory of Equations.

The course entitled "Review of Engineering Mathematics," taught by C. G. Stipe and D. R. Davis, was open only to students who had completed Calculus. In forty lessons the subjects of Algebra, Trigonometry, Analytic Geometry, and Calculus were reviewed.

Mathematics was a popular subject at Shrivenham. The course registrations were approximately 700 during each of the two terms. In number of registrations, Mathematics was second to Business Administration, and was ahead of English and Foreign Languages, which followed in that order. At Shrivenham there was a free elective system, and mathematics prospered rather than suffered under such a system.

4. *The Shrivenham Mathematics Colloquium.* The Army university centers at Biarritz and Florence have already been described in detail,* and hence this paper is restricted essentially to activities that took place at Shrivenham.

Soon after our arrival at Shrivenham, it was suggested that the mathematics group hold meetings for the presentation of scientific papers. Accordingly on Tuesday, September 4, there was held the first meeting of the Shrivenham Mathematics Colloquium. The speaker at that meeting was C. G. Latimer, his topic being: "The Representation of an Integer as the Sum of Four Squares." Meetings of the Colloquium were held approximately every two weeks. The succeeding speakers were P. R. Rider: "The Analysis of Variance"; J. P. Ballantine: "Triads of Arccotangents"; E. T. Browne: "Concerning a Certain Ring of Homographies." At the next meeting, P. D. Edwards was in charge of a showing of British moving picture films of a mathematical nature. The final meeting on November 27, was addressed by T. C. Benton on the topic: "The Flow of a Perfect Fluid over a Right-Angled Step."

There was a good attendance at all meetings of the Colloquium. Besides the members of the department, there were a few advanced students of mathematics and members of other departments who were regular attendants at the meetings.

5. *Relations of Shrivenham with Oxford University.* In connection with the course in Astronomy at Shrivenham, an attempt was made to arrange a visit of students and faculty to the Greenwich Observatory, but permission for the visit could not be obtained. This was exceptional, because in general the faculty and students of Shrivenham were welcome at all British scientific and educational institutions.

* T. A. Bancroft and R. M. Winger: Mathematics at the American Universities in Europe, this MONTHLY, vol. 53, 1946, pp. 131–135.

Oxford was only twenty miles away and there were frequent contacts between the mathematicians of the two institutions. This was due chiefly to the efforts of T. W. Chaundy of Oxford. Through him the facilities of Oxford University were made available to those who wished to use them. I made frequent visits to the Oxford libraries while preparing the lectures that I delivered at Amsterdam University in December.

On October 26, the members of the Shrivenham Mathematics Department attended the 279th meeting of the Oxford University Mathematical and Physical Society. Professor Whitehead presided and Professor Milne spoke on "The Conservation of Momentum." At a later meeting of the Oxford Society, D. R. Davis and E. T. Browne were the speakers.

6. *Closing of the Army Universities.* The unexpected collapse of Japanese military resistance in August, 1945, changed the plans of the Army for the continued operation of the Army universities. Additional ships were now available for sending men home from Europe. It was decided to concentrate all American troops in Germany and to discontinue the schools in England and in France. Hence, instead of the five two-month terms that were originally planned, Shrivenham was closed at the end of the second term in early December. The university at Biarritz was continued for a third term with half the normal enrollment, and was finally closed on March 15, 1946.

The end of the Japanese war and the resulting influx of students into American colleges in September, 1945, made many faculty members of the Army universities anxious to return to their colleges at the earliest possible date. A number returned to this country in December and January in time to take up their usual teaching duties with the opening of the second semester of 1945–46. Those who wished to carry out the original plans and spend the entire year with the Army were sent to various points in Germany after the universities at Shrivenham and Biarritz had closed. There they were assigned to various educational projects among the troops in the army of occupation.

7. *The Army Education Program in Germany.** During the winter and spring of 1946, army schools giving courses in mathematics of college grade were in operation in Vienna and in the Bremen area. Of the schools conducted at this time by the occupation forces, the most ambitious was the Weihenstephen Agricultural and Technical College conducted by the Third Army at Freising, near Munich. The college was located in buildings erected for a branch of the University of Munich. Courses were given in College Algebra, Plane and Spherical Trigonometry, Analytic Geometry, and Calculus. In addition, Professor Lowenstein conducted an advanced class in the Theory of Numbers.

One characteristic of Weihenstephen was that students studied only one subject. Classes met four hours per day for five days per week, the twenty hours weekly being divided into periods of regular instruction and supervised study. Each course continued for a term of eight weeks.

After April 1, 1946, all courses upon the college level were discontinued. The

* I am indebted to L. L. Lowenstein and P. D. Edwards for information on this program.

schools which continued in operation after that date offered only secondary school subjects.

8. *The Veteran and the Veterans Problem.* Veterans are now presenting many problems to the colleges. While I cannot claim to be an expert on all aspects of the subject of veterans, I and my colleagues in Europe saw these men when they were making their first adjustments from a life of combat to a life of study.

During the first few weeks of the first term at Shrivenham, the Mathematics Office was crowded at all hours of the day (including Saturday morning and afternoon) with soldier students who wished assistance in the preparation of their lessons. During their years of combat duty, many had forgotten how to study, to say nothing of having forgotten the most elementary aspects of mathematics. It was noticeable that at the beginning of the second term there was much less demand for assistance of this sort. During the intervening two months, the soldiers who came to us in the second term had been busy with occupation duty rather than combat duty and consequently had had more leisure time to devote to reading and study. These men had become accustomed to a non-combat type of army life, which is one step toward making an adjustment to civilian life.

In the same way, the present veteran student comes to college either after an interval of civilian life or, if he comes directly from one of the armed services, after a period of non-combat service. In either case he has been able to look forward to college life, has been able to plan for it over a period of time, and if he is wise, has been trying to accustom himself to habits of study. For this reason I believe that the necessity for giving refresher courses has largely passed, although local conditions may make them necessary in some colleges.

In general, the veteran is a normal, well-adjusted person. Travel and contact with persons of various nationalities have given him the ability to adjust easily to new situations. He is serious, earnest, and well-motivated. Many colleges report that the veterans are superior in grades, in reliability, and in studiousness to the general run of students. Often the veteran shows a lack of interest in college life, in social activities and particularly in inter-collegiate athletics. He is interested in finishing college as soon as possible and hence has little time and energy for extra-curricular activities. This is especially true in the case of the veteran with dependents. Because the typical veteran feels the need for making up some of the time he has lost, the increased summer school registration of the past year will probably continue for several years. The veteran feels the necessity for accelerated courses, whether or not the colleges feel that they are educationally desirable.

The great majority of veterans are interested only in professional courses: engineering, science, business, law, medicine, *etc.* In the pursuit of such courses, veterans will take many courses in mathematics. Hence it is safe to predict increased registration in mathematics for the next few years.

I hope that departments of mathematics will not be too rigid in enforcing prerequisites for courses. For example, a veteran may have learned sufficient

trigonometry in his training in navigation or gunnery so that he is prepared for the study of analytic geometry, although no formal credit for a course in trigonometry may appear on his academic record. Compelling him to repeat trigonometry may dampen his ardor for the study of mathematics.

I believe that in spite of the great number of students now in college, faculty members will find that the veterans are anxious to learn, and that many of them are eager to continue their study of mathematics. If the latter point can be kept in mind, we may be able to forget some of our troubles.

THE PREDOCTORAL FELLOWSHIPS

The Rockefeller Foundation made a grant of $335,000 to the National Research Council in the autumn of 1944 to provide predoctoral fellowships in the natural sciences for the benefit of students whose studies were interrupted by war activities. A committee consisting of the following persons was appointed to administer the fellowships: John T. Tate, research professor of physics, University of Minnesota (chairman); Henry A. Barton, director, American Institute of Physics; Charles W. Bray, professor of psychology, Princeton University; Detlev W. Bronk, professor of biophysics, University of Pennsylvania; Luther P. Eisenhart, dean and professor emeritus of mathematics, Princeton University; and W. Albert Noyes, Jr., professor of chemistry, University of Rochester. In January, 1946, Dr. Noyes succeeded Dr. Tate as chairman.

The Committee began its deliberations in June 1945, and applications were first received during the fall of 1945. In a very short time it became apparent that the number of applications would far exceed predictions, and the Rockefeller Foundation subsequently made a grant of $250,000 in addition to the amount originally given.

At each meeting of the committee, awards were apportioned generally according to the number of applications reviewed in each science, but at the final meeting the number awarded was such as to make the distribution among the various sciences approximately proportional to the final percentage of applications in each science, so as to assure the development of men in all fields.

The following table gives the number of applications and the number of awards in each science:

Science	*Applications*	*Awards*
Biology	266	20
Chemistry	733	61
Geology	127	9
Mathematics	207	18
Metallurgy	14	3
Physics	841	75
Psychology	114	10
Total	2,302	196

The stipends are $1,200 per year for single men and $1,800 per year for married men. In addition, an amount up to $500 per year is allowed for tuition and

fees. For a man who receives benefits under the G. I. Bill of Rights the stipend is such as to bring his total income to the appropriate one of the above figures; that is, $420 additional is awarded a single man and $720 additional is given a married man.

Since all awards of fellowships are made for one year and renewal is contingent upon evidence of satisfactory work, the Committee will maintain records of the recipients and receive periodic official reports from the institutions where the men are enrolled.

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE ANNUAL MEETING OF THE KENTUCKY SECTION

The twenty-eighth annual meeting of the Kentucky Section of the Mathematical Association of America was held at the University of Louisville, Louisville, Kentucky, on Saturday, April 27, 1946, in conjunction with the annual meeting of the Kentucky Academy of Science. Dr. Aughtum S. Howard, Chairman of the Section, presided.

There were fifty-nine in attendance, including the following sixteen members of the Association: M. C. Brown, J. C. Eaves, H. P. Fawcett, W. L. Fields, Clarence Ford, Charles Hatfield, Sr., Aughtum S. Howard, W. R. Hutcherson, W. L. Moore, Sister Charles Mary Morrison, S. T. Parker, D. W. Pugsley, W. J. Robinson, J. H. Simester, D. E. South, Guy Stevenson.

At the business meeting the following officers were elected for the coming year: Chairman, Guy Stevenson, University of Louisville; Secretary, M. C. Brown, University of Kentucky.

The following program was presented:

1. *An interesting approach to analytic geometry*, by Dr. W. L. Moore, University of Louisville.

It was pointed out that since analytic geometry is a correspondence between algebra and geometry, and since similar correspondences exist between concepts and science and algebra, then more experience with coördinate systems should be offered to the student. In addition to the rectangular, oblique, and polar coördinate systems, some attention should be given to the theory of algebra of points as outlined in Veblen and Young, and in the Carus Monograph on projective geometry by Young. A brief outline of the basic concepts was then given. Dr. Moore felt that these concepts could be understood by advanced undergraduates.

2. *Color reproduction matrices*, by Sister Mary Charlotte Fowler, Nazareth College, introduced by Sister Charles Mary Morrison.

Matrix algebra was applied to the gelatin matrices of a color print. From a set of equations representing the red, green and blue densities of the reproduction, a color matrix was obtained in which for a set of ideal primaries the elements on the main diagonal were unity, all others zero. In order to reproduce grays correctly, it is often necessary to change these dye concentrations, and this is done by means of the inverse color matrix operating on the vector densities of the reproduction. As a result of this operation, another set of nine coefficients was obtained, and these form the neutralized color matrix. In this matrix the elements on the main diagonal represent the gammas to which the three color elements must be printed to render gray correctly.

3. *Perturbation theory in wave mechanics*, by Professor W. L. Fields, Louisville Municipal College.

The subject of wave mechanics concerns all questions relative to electrons, atoms, systems of electrons and atomic nuclei which lead to the study of Schroedinger's equation, which is

$$\sum \frac{1}{m_k}\left(\frac{\partial^2\psi}{\partial x_k^2}+\frac{\partial^2\psi}{\partial y_k^2}+\frac{\partial^2\psi}{\partial z_k^2}\right)+\frac{8\pi^2}{h^2}(E-V)\psi=0.$$

The solution of this equation cannot be performed rigorously in every case. It is often necessary to resort to perturbations, or other approximate methods for solutions. A small perturbation such as an electrical or magnetic field will change the variables in the equation by amounts which can readily be computed.

4. *Post war mathematics on a college campus*, by Dr. W. R. Hutcherson, Berea College.

On the Berea College campus much recent attention has been devoted to remedial mathematics for the weaker students in the regular mathematics courses. A public mathematical exhibit staged by mathematics majors and the quickened interest in mathematics on the part of veterans were discussed.

5. *The relationship of mathematics to Christianity*, by Professor Beulah Graham, Campbellsville College, introduced by M. C. Brown.

The speaker discussed the relation between mathematical and religious thought.

6. *Some recursion formulas*, by Professor S. T. Parker, University of Louisville.

Certain recursion relations, developed with the aid of Dean W. H. Durfee of Hobart College, were presented. Applications to the solution of Diophantine equations were explained.

7. *The education of mathematics teachers*, by Dr. H. P. Fawcett, Ohio State University.

The speaker cited various qualities which the teacher of mathematics should possess. It was remarked that, in addition to competence in the field of mathe-

matics, the teacher should have a broad cultural background. Courses in the history of mathematics, in fundamental concepts, and in applications, were recommended for teachers, as were other details of teacher training.

8. *Stirling's formula for factorial N*, by Dr. D. E. South, University of Kentucky.

A development of Stirling's formula based upon the well known methods and notations of the calculus of finite differences was discussed. This development had been written to be used in a physics course in which the students had studied no mathematics beyond the integral calculus.

M. C. BROWN, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirtieth Annual Meeting, Swarthmore, Pa., December 26–27, 1946.

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ILLINOIS, Peoria, May 9–10, 1947
INDIANA
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KANSAS
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MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Baltimore, Md., December 7, 1946
METROPOLITAN NEW YORK
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NEBRASKA, Lincoln, May 3, 1947
NORTHERN CALIFORNIA, San Francisco, January 25, 1947
OHIO, Columbus, April 3, 1947
OKLAHOMA
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SOUTHEASTERN, Columbia, S. C., April, 1947
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Edited by OTTO DUNKEL, Washington University, ORRIN FRINK, JR., Pennsylvania State College, and HOWARD EVES, Oregon State College.

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